Experimental Observation of Multifractality in Fibonacci Chains

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arXiv:2207.13755

From the Fibonacci Sequence to Fibonacci Chains

Fibonacci numbers:

0,1,1,2,3,5,8,13,21,34,55,89,…

$$F_{n+1} = F_n + F_{n-1}$$

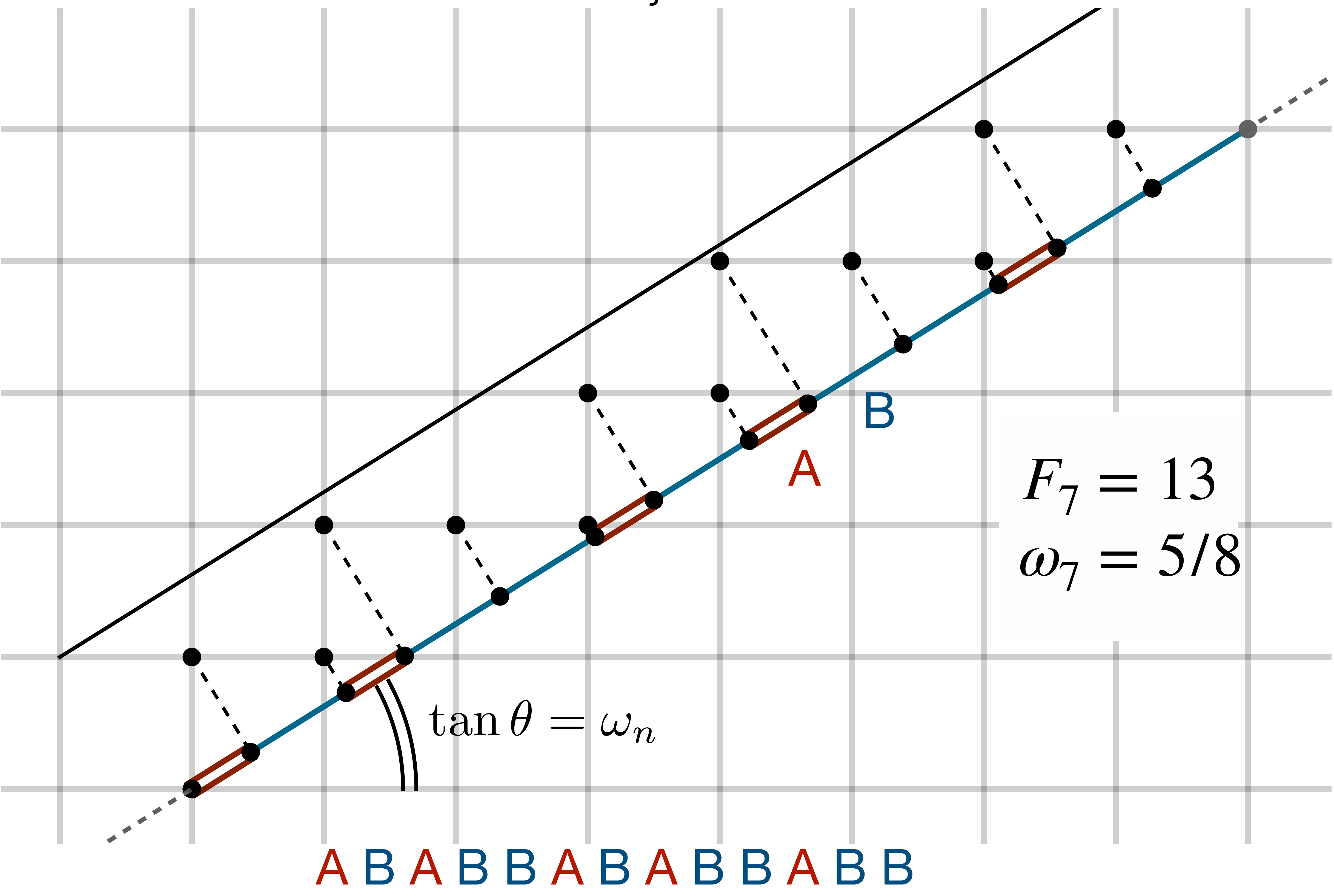
$$\omega_n = \frac{F_{n-2}}{F_{n-1}}$$

Fibonacci sequence:

$$\omega_n \xrightarrow[n\to\infty]{} \omega = \frac{\sqrt{5}-1}{2}$$

golden mean

Cut & Project



- If $\tan\theta$ is rational

—> periodic sequence with motif length F_n

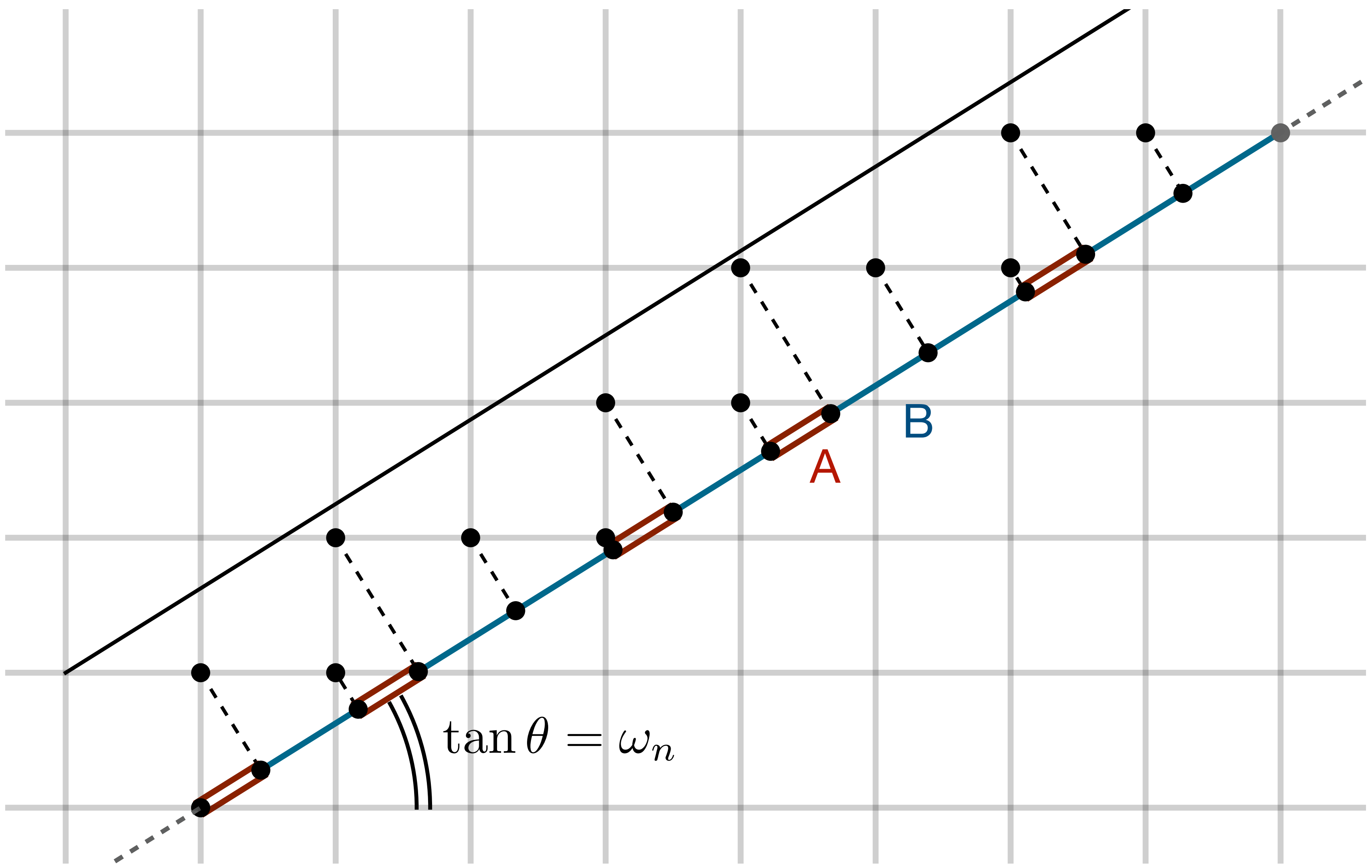
- $n \to \infty$, $\tan\theta$ becomes irrational

—> quasi-periodic sequence

The Off-diagonal Fibonacci Model:

- Tight-Binding Hamiltonian H with constant on-site energies ν_0
- Assign coupling strength t_A to letter A, t_B to the letter B.
- Hopping constants follow the Fibonacci Sequence
- Important parameter $\rho = t_B / t_A$

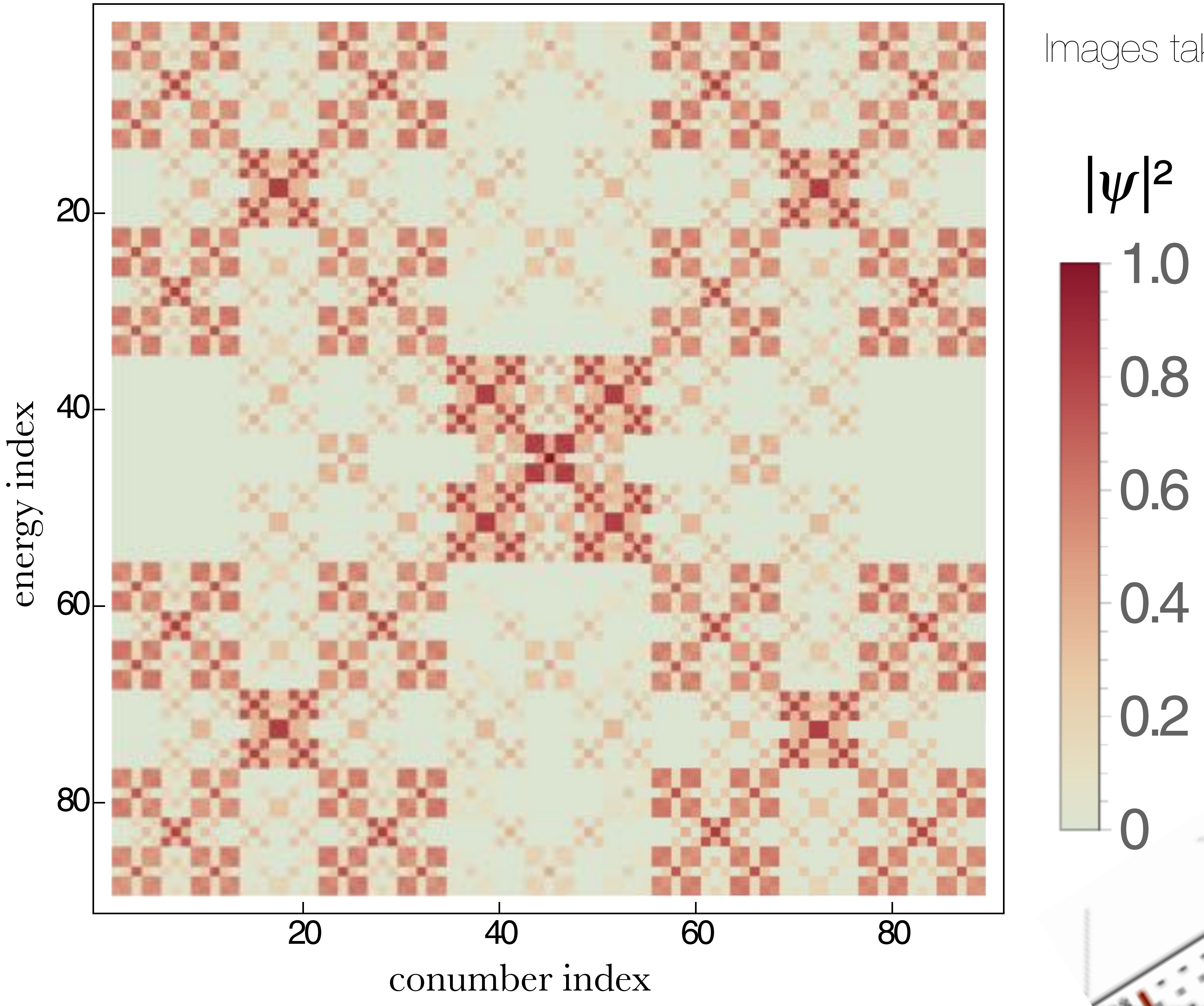
Cut & Project



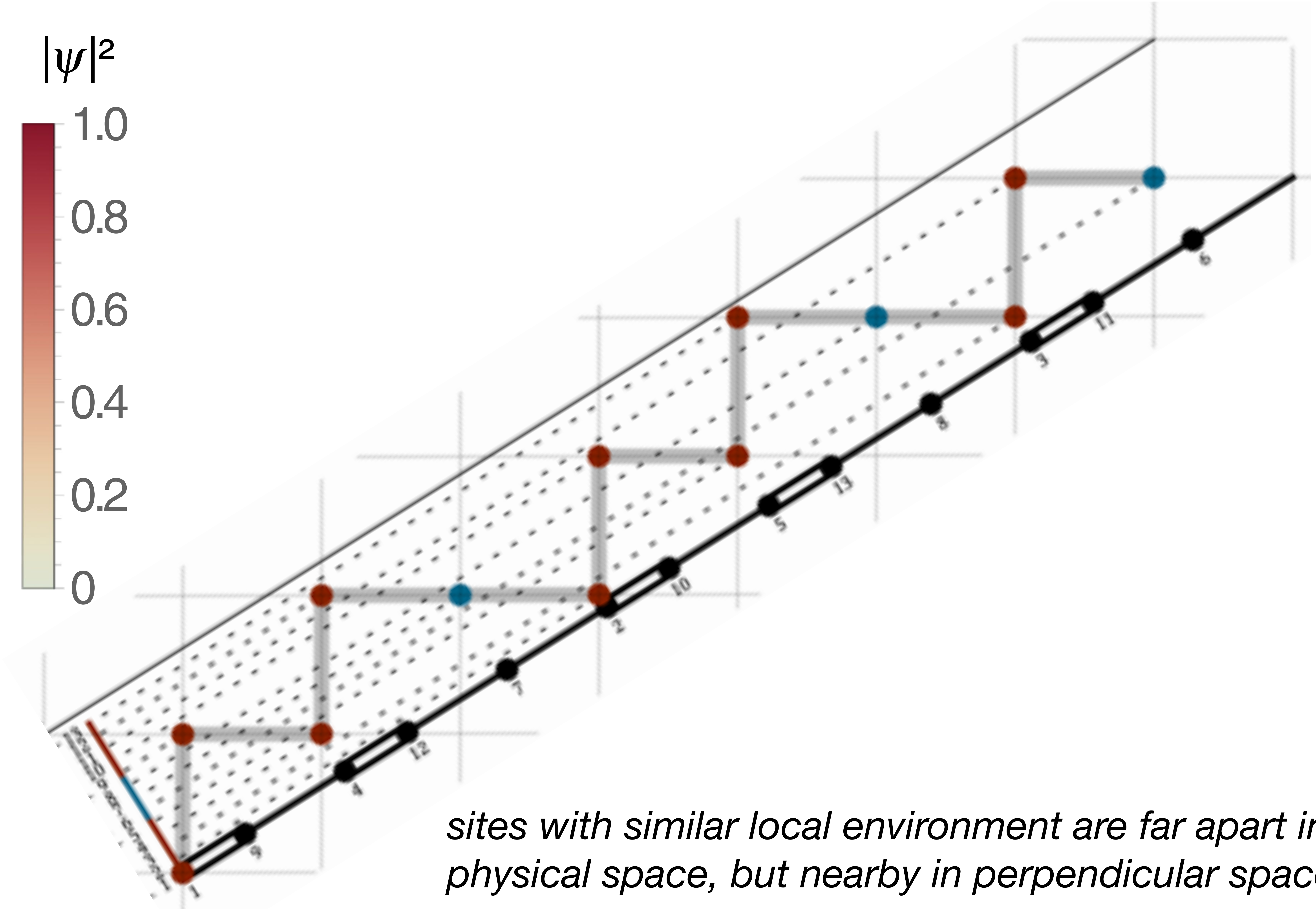
$$H = \begin{pmatrix} \nu_0 & t_A & 0 & 0 & 0 & 0 & 0 & \dots \\ t_A & \nu_0 & t_B & 0 & 0 & 0 & 0 & \dots \\ 0 & t_B & \nu_0 & t_A & 0 & 0 & 0 & \dots \\ 0 & 0 & t_A & \nu_0 & t_B & 0 & 0 & \dots \\ 0 & 0 & 0 & t_B & \nu_0 & t_B & 0 & \dots \\ 0 & 0 & 0 & 0 & t_B & \nu_0 & t_A & \dots \\ 0 & 0 & 0 & 0 & 0 & t_A & \nu_0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

—> Check for Fractality in eigenvalues and eigenstates

The Fibonacci Chain — a well known 1D Quasicrystal



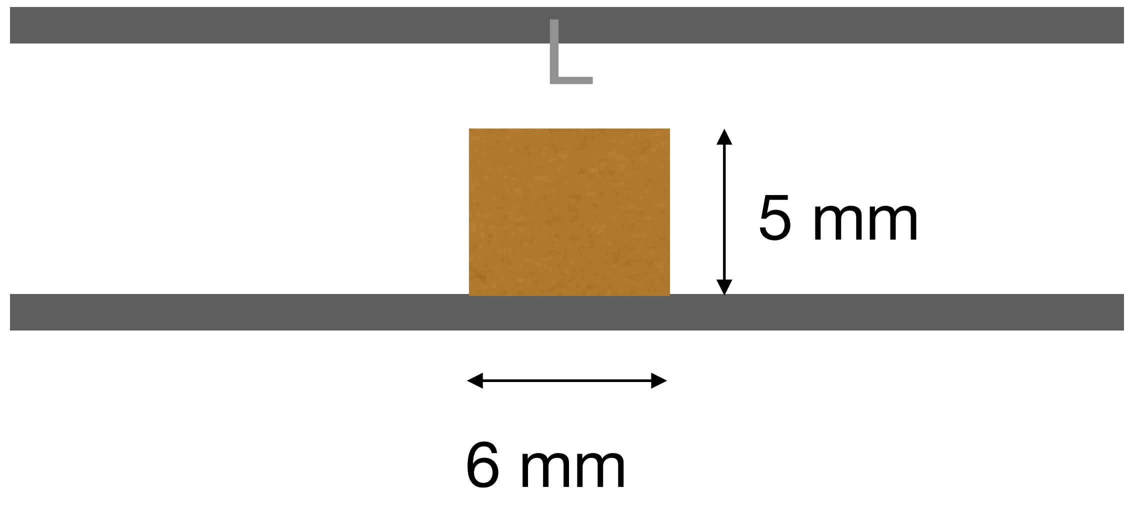
Images taken from *Macé et al., PHYSICAL REVIEW B 93, 205153 (2016)*

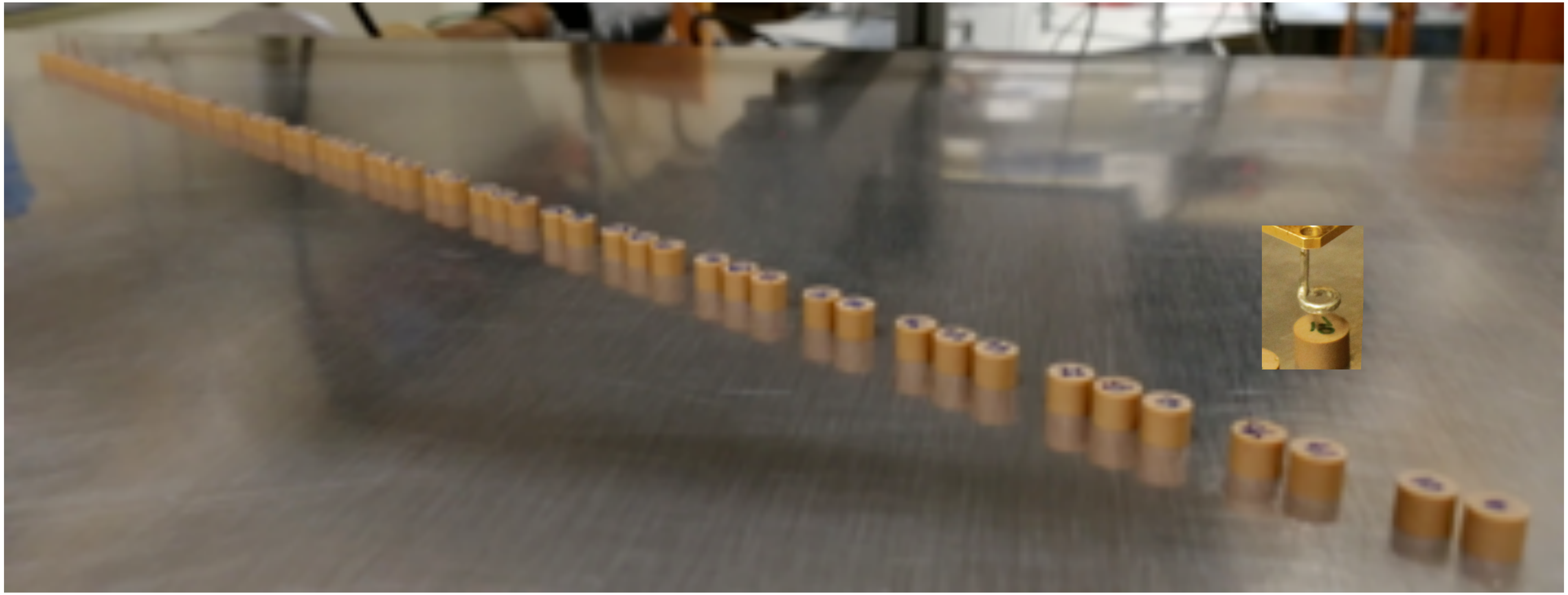


sites with similar local environment are far apart in physical space, but nearby in perpendicular space

The Experimental Fibonacci Chain

- coupled dielectric resonators, $\nu_0 \approx 7.45$ GHz
- evanescent coupling —> $d_A = 7$ mm ($t_A = 126$ MHz) and $d_B = 8$ mm ($t_B = 81$ MHz)
- measure the reflection spectrum above each resonator —> local density of states





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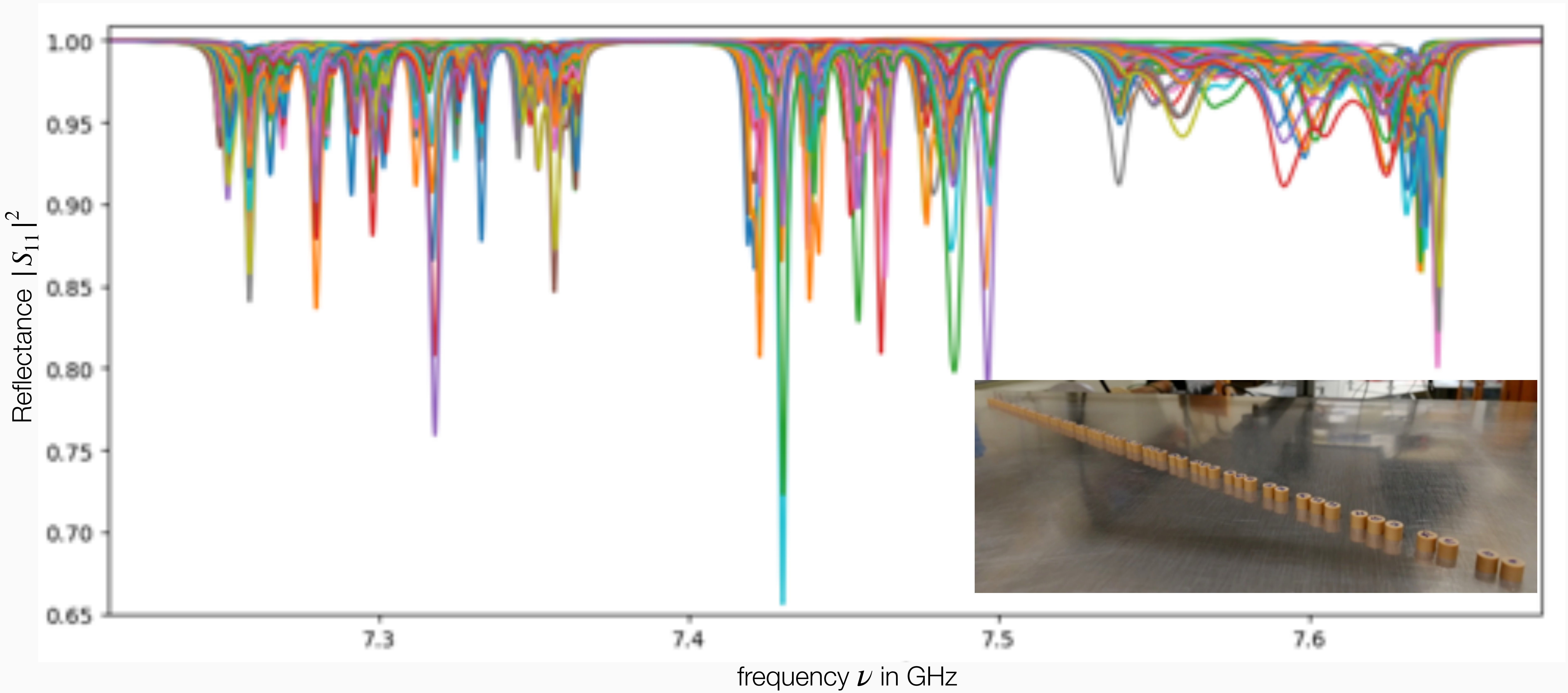
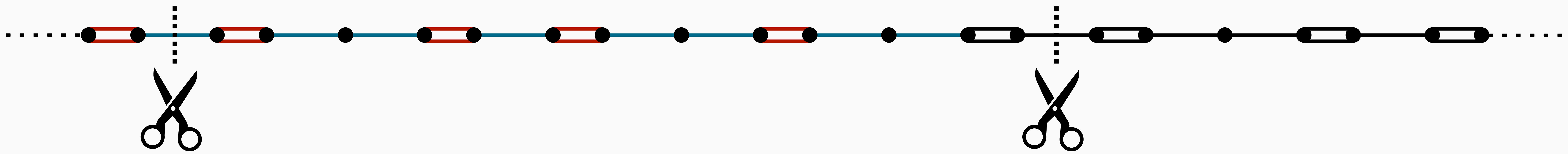


- measure S_{11} above all sites
- extract complex resonance frequencies and amplitudes for all peaks for all positions
- acces to all eigenvalues $\{\nu_j\}$ and eigenstates $\{\Psi_j(i)\}$

Average over Different Permutations

- one basic motif ($F_7 = 13$)
- Different permutations of basic motif ($F_7 = 13$)
- “cut” on (neglect) a weak coupling
- chains end on strong coupling —> no edge states

Average over Different Permutations



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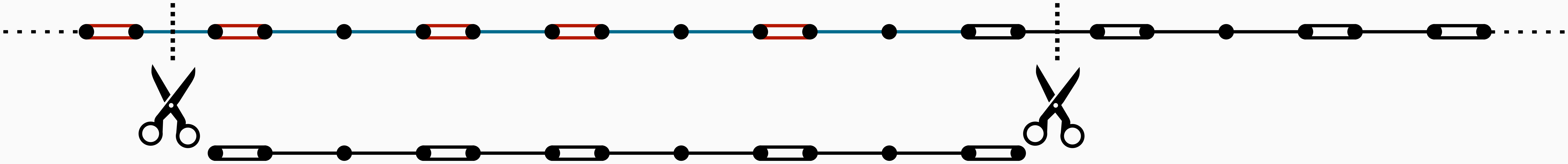

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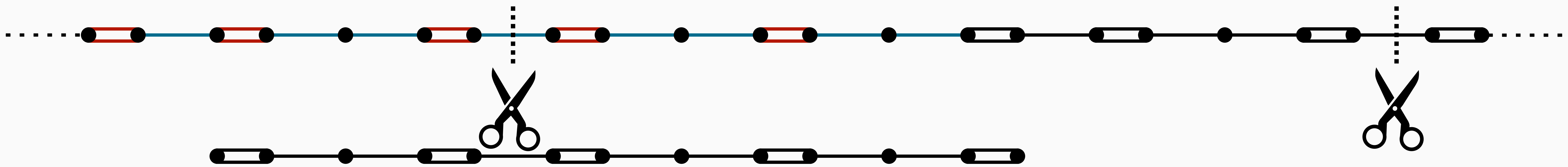

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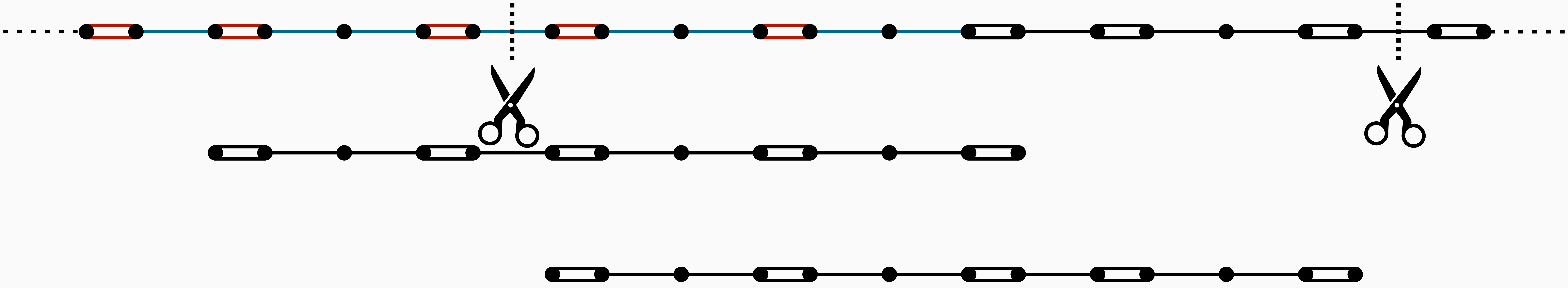

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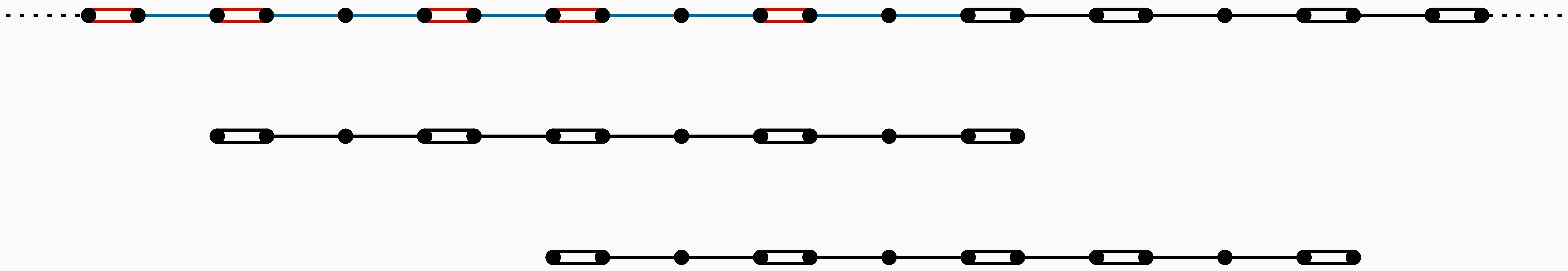

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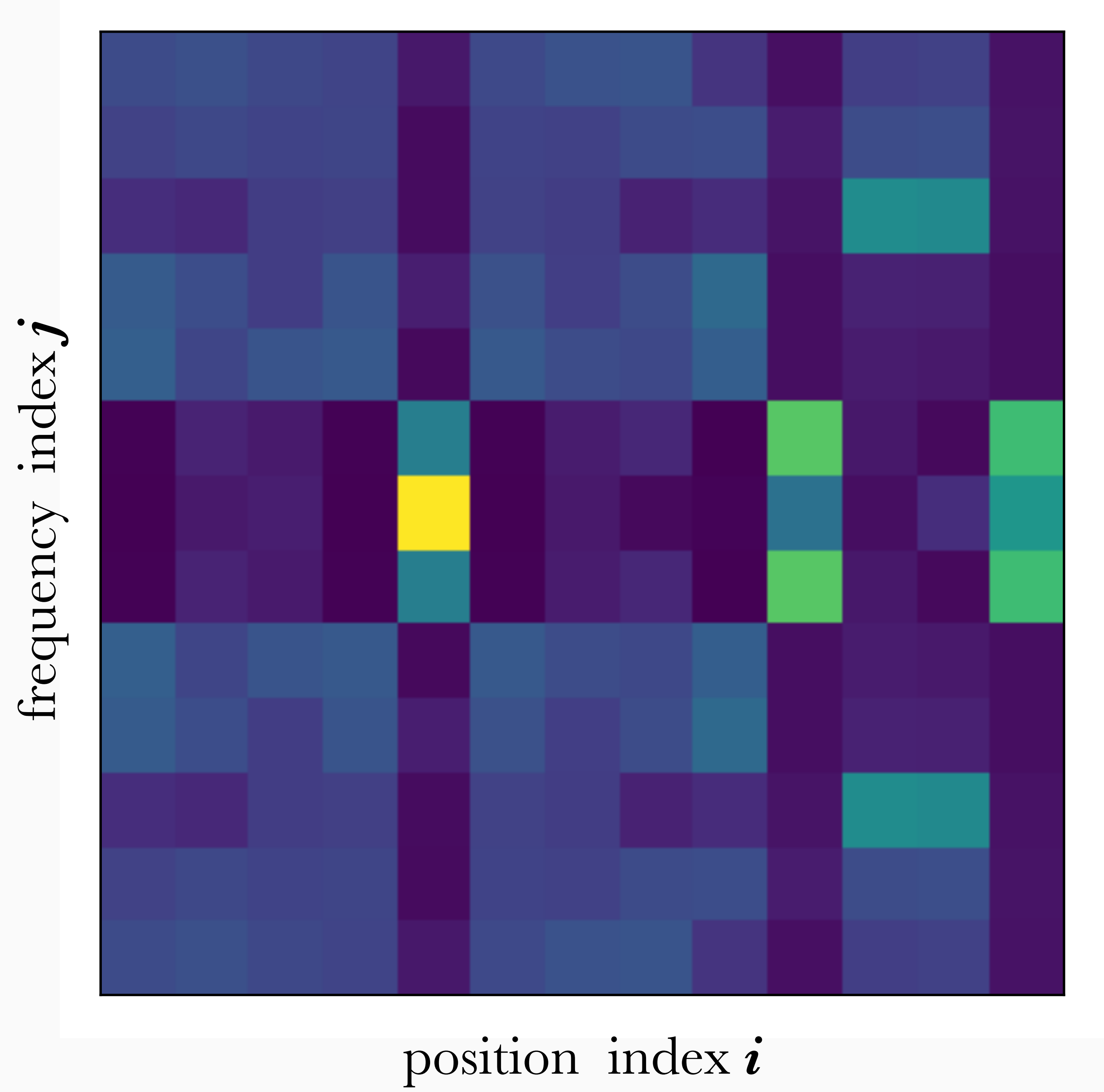
Average over Different Permutations

- $F_{10} = 55$ resonators
- 8 different permutations

Revealing the Fractality via Conumbers

$F_7 = 13$



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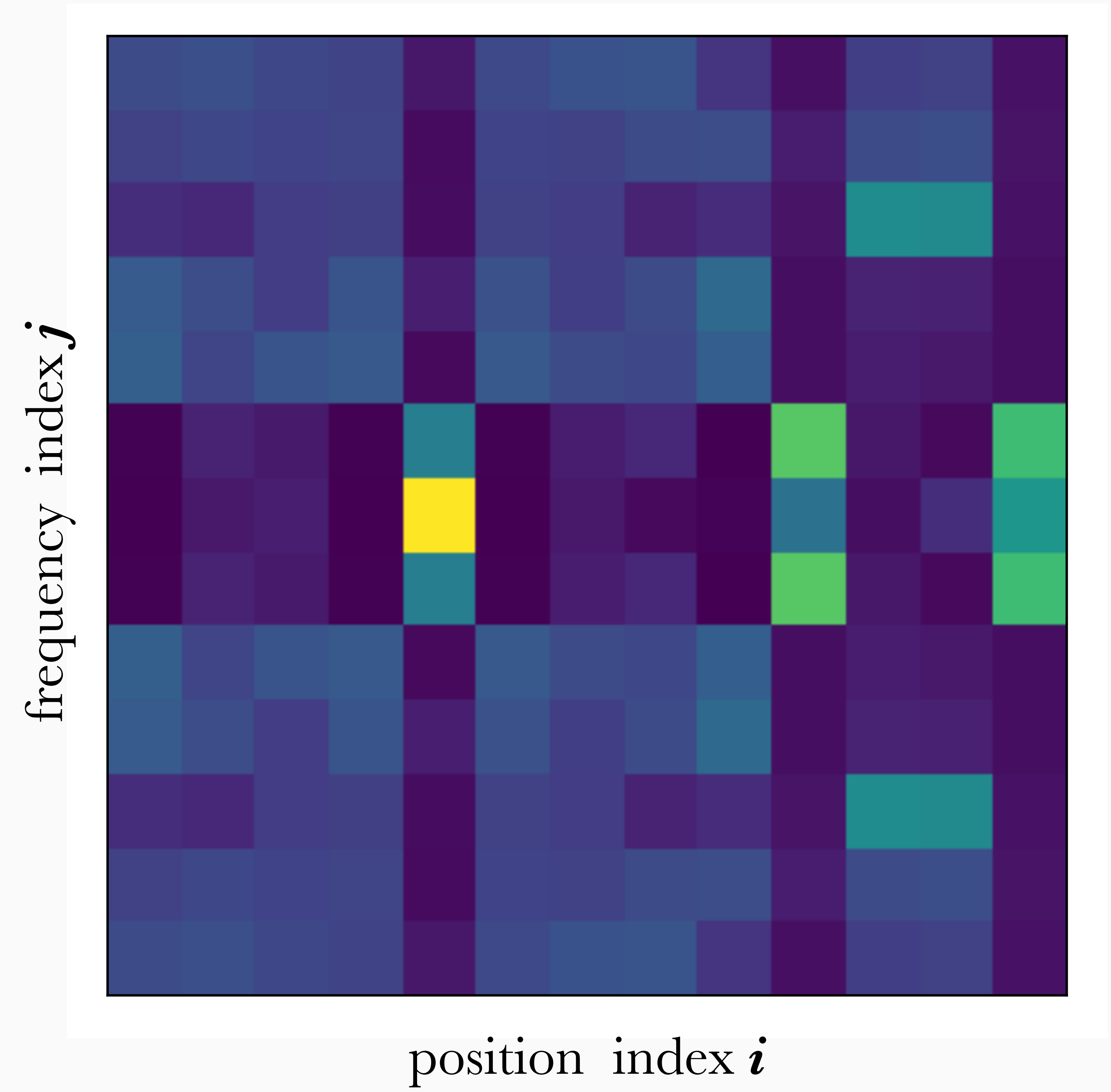

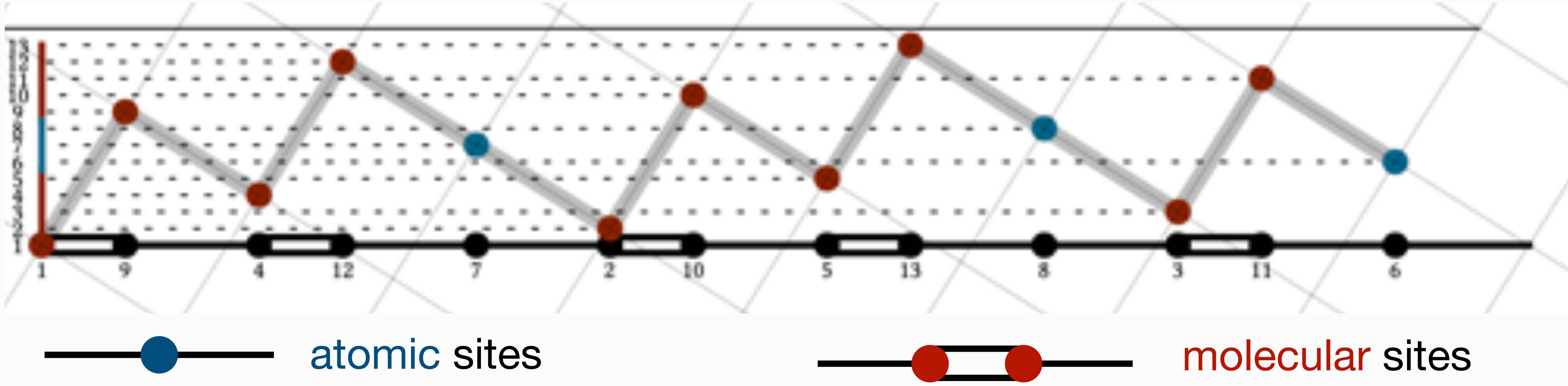

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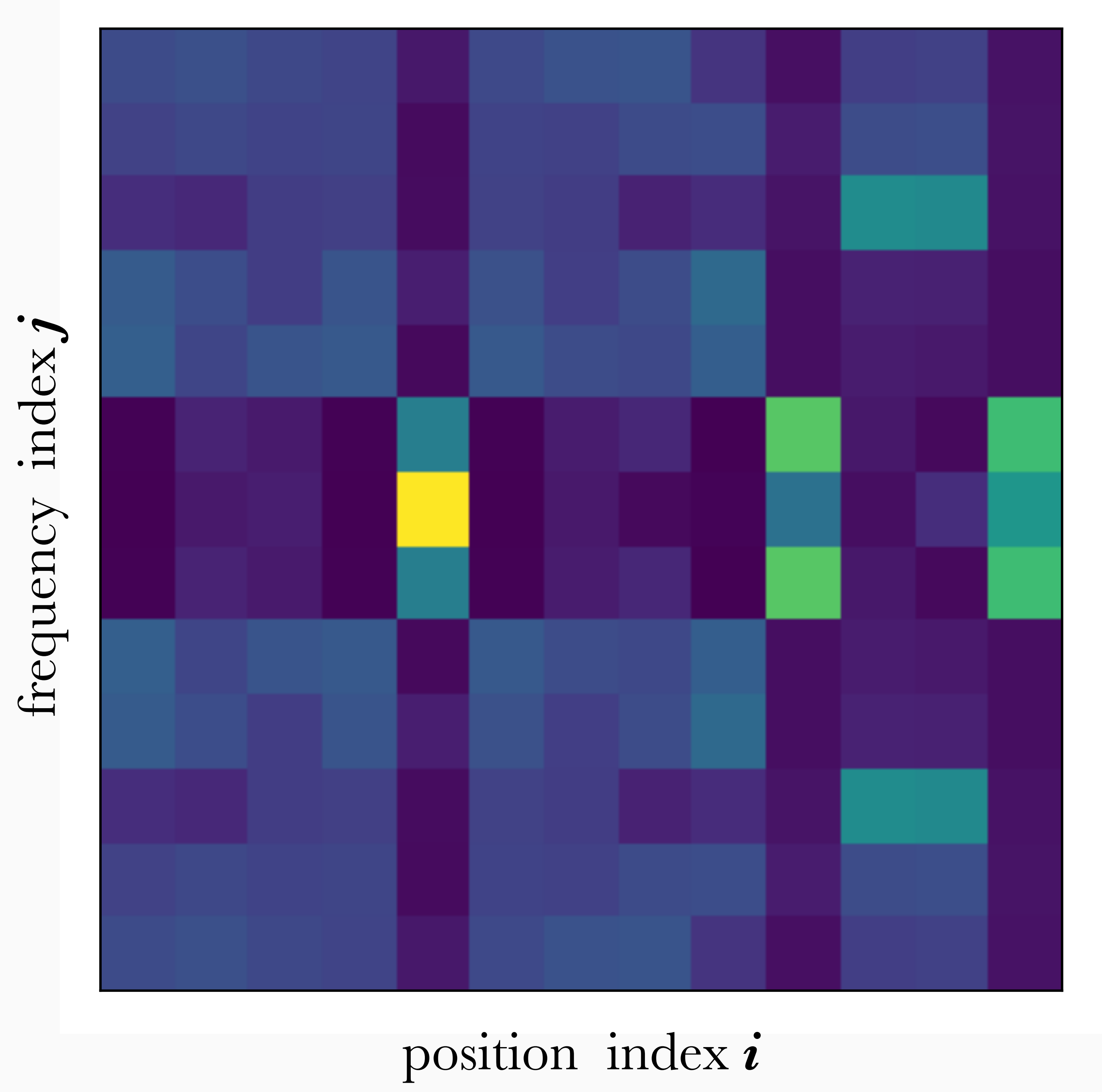
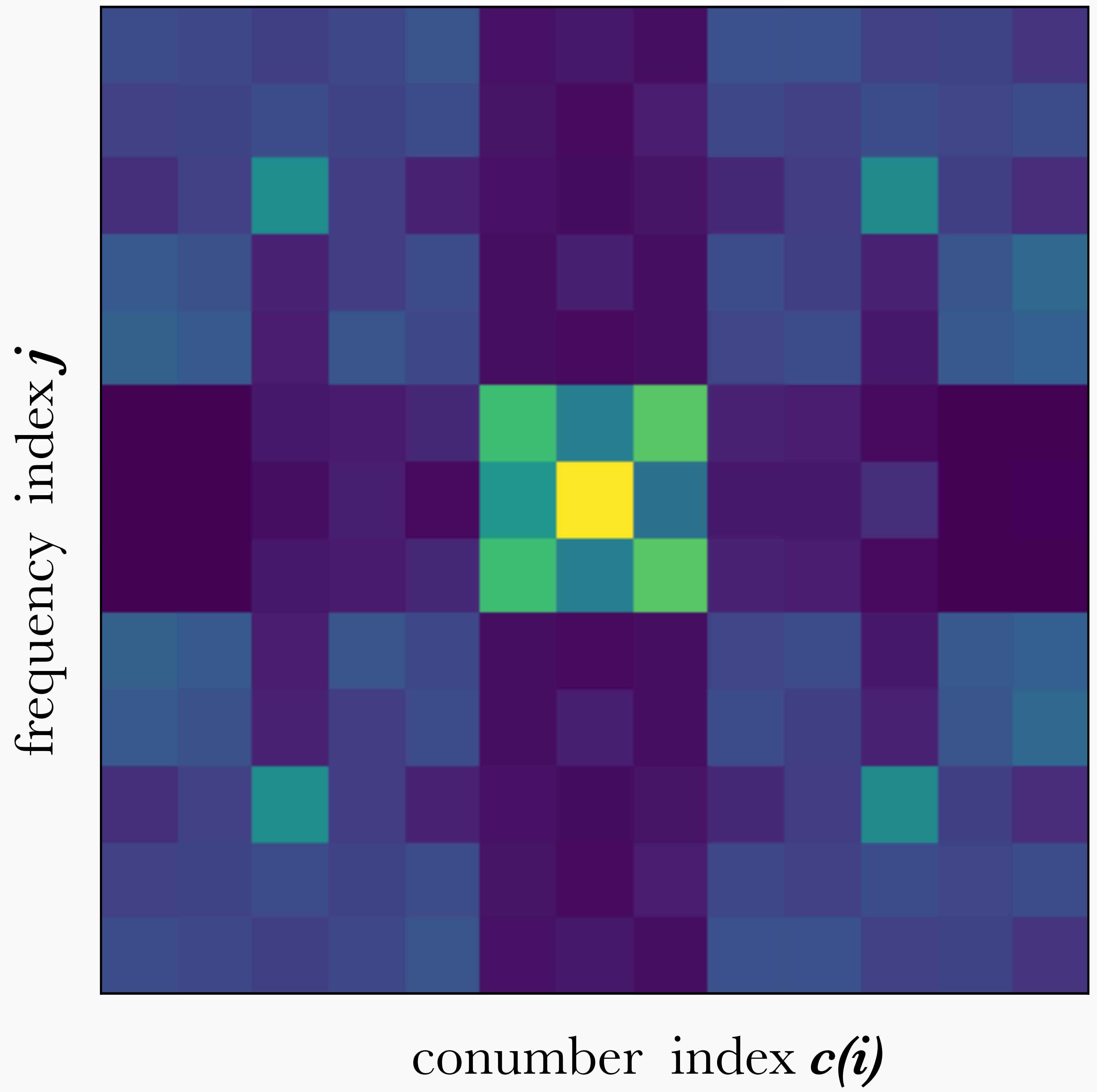
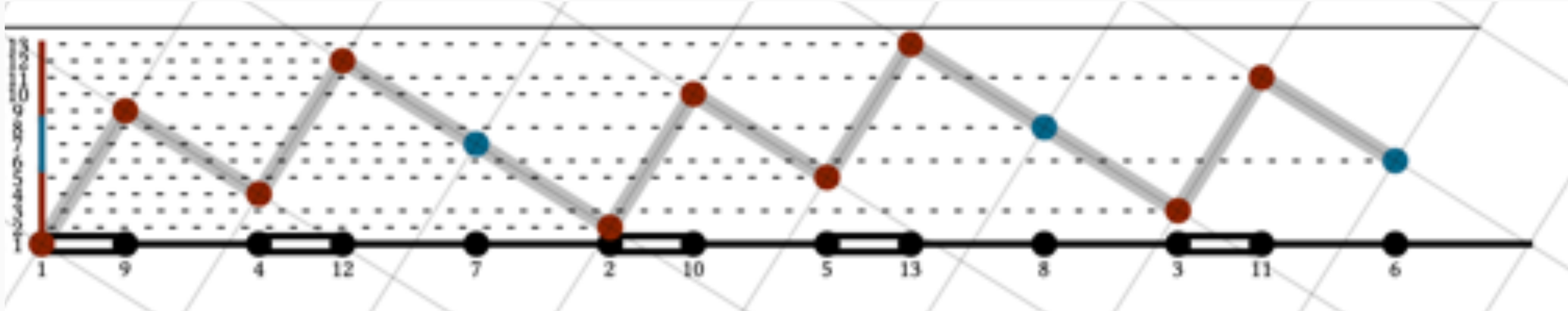






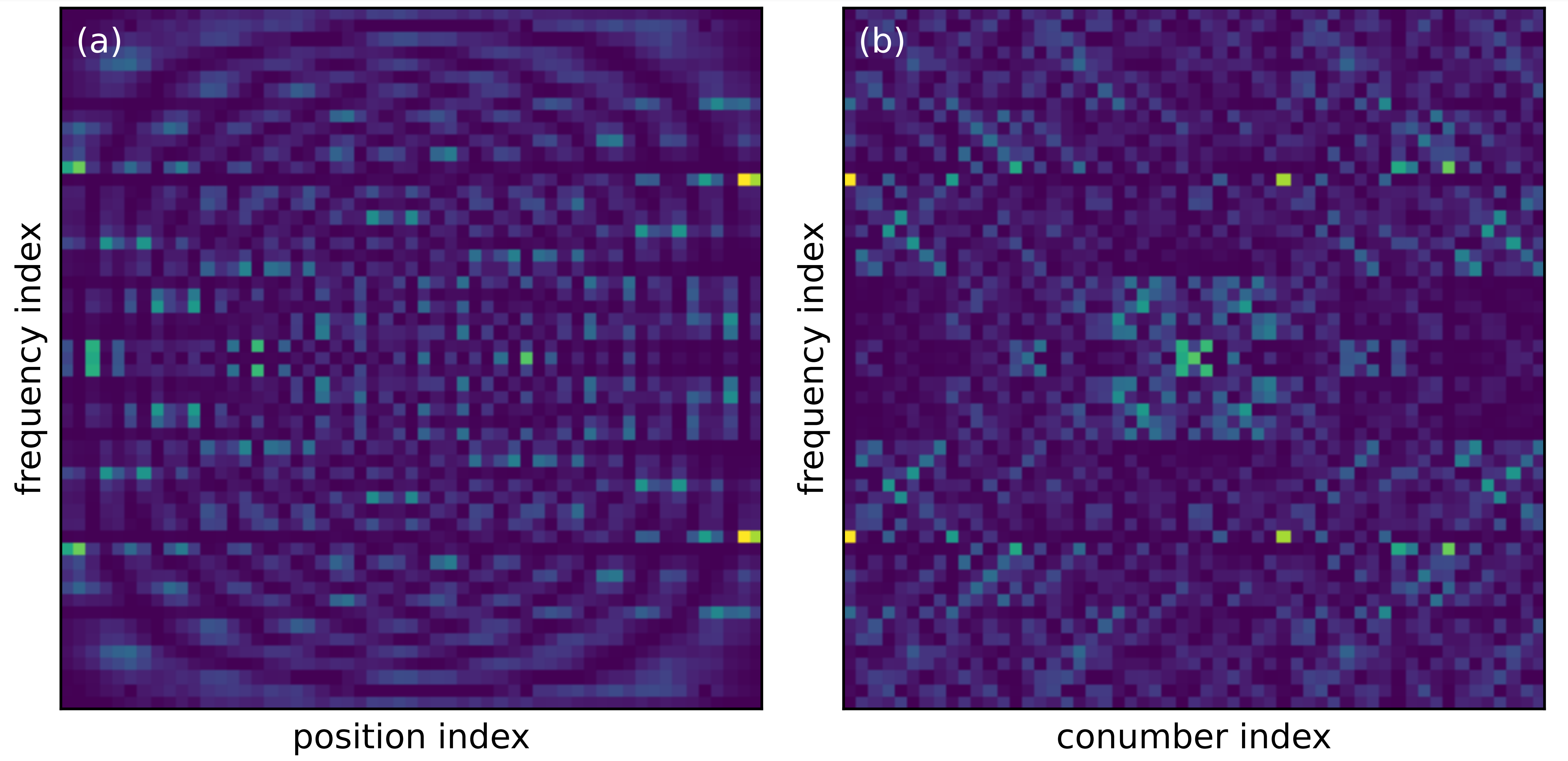
atomic sites

molecular sites

Revealing the Fractality via Conumbers

$$F_7 = 55$$

one single configuration

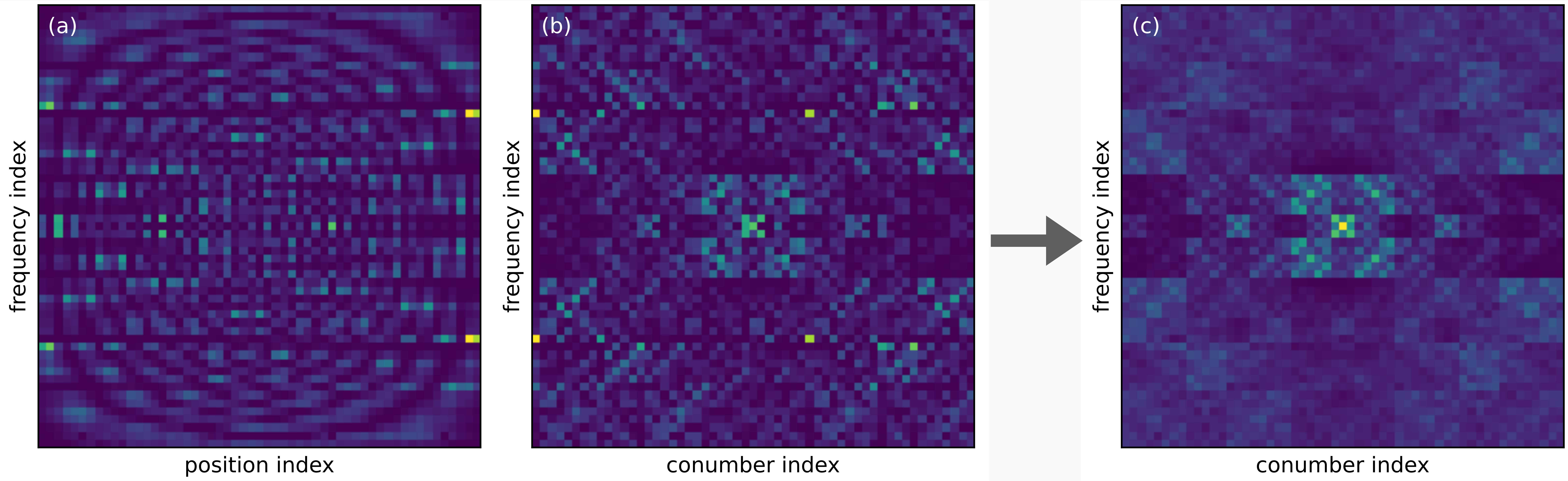


Revealing the Fractality via Conumbers

$F_7 = 55$

one single configuration

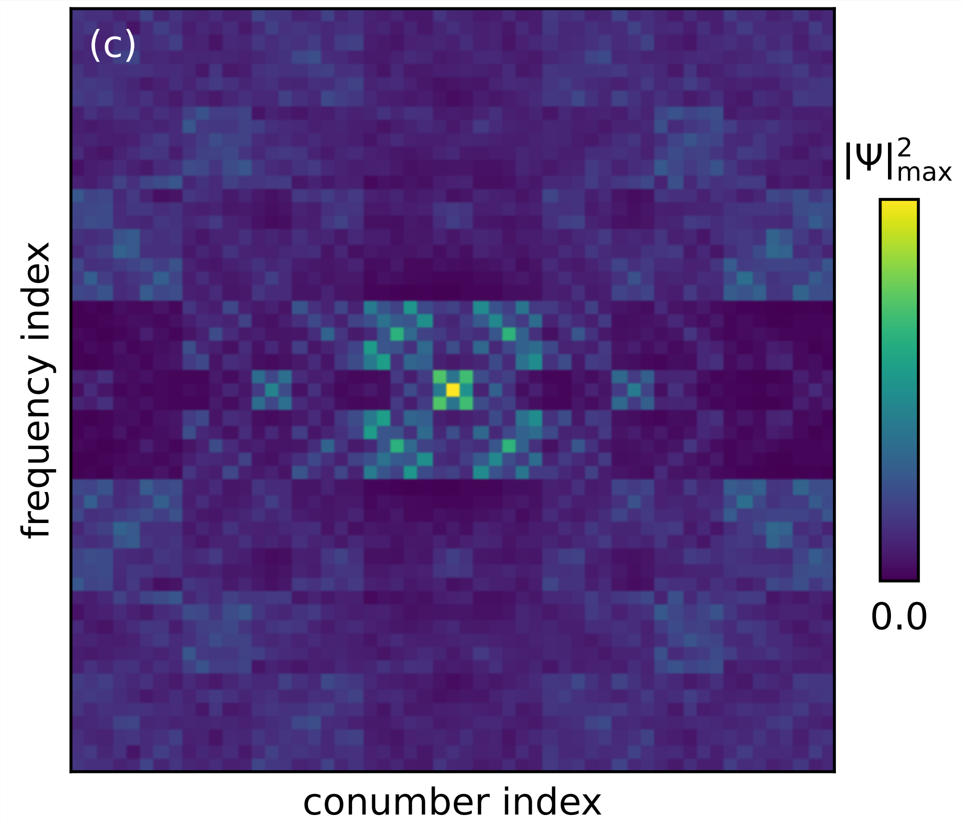
average over all permutations



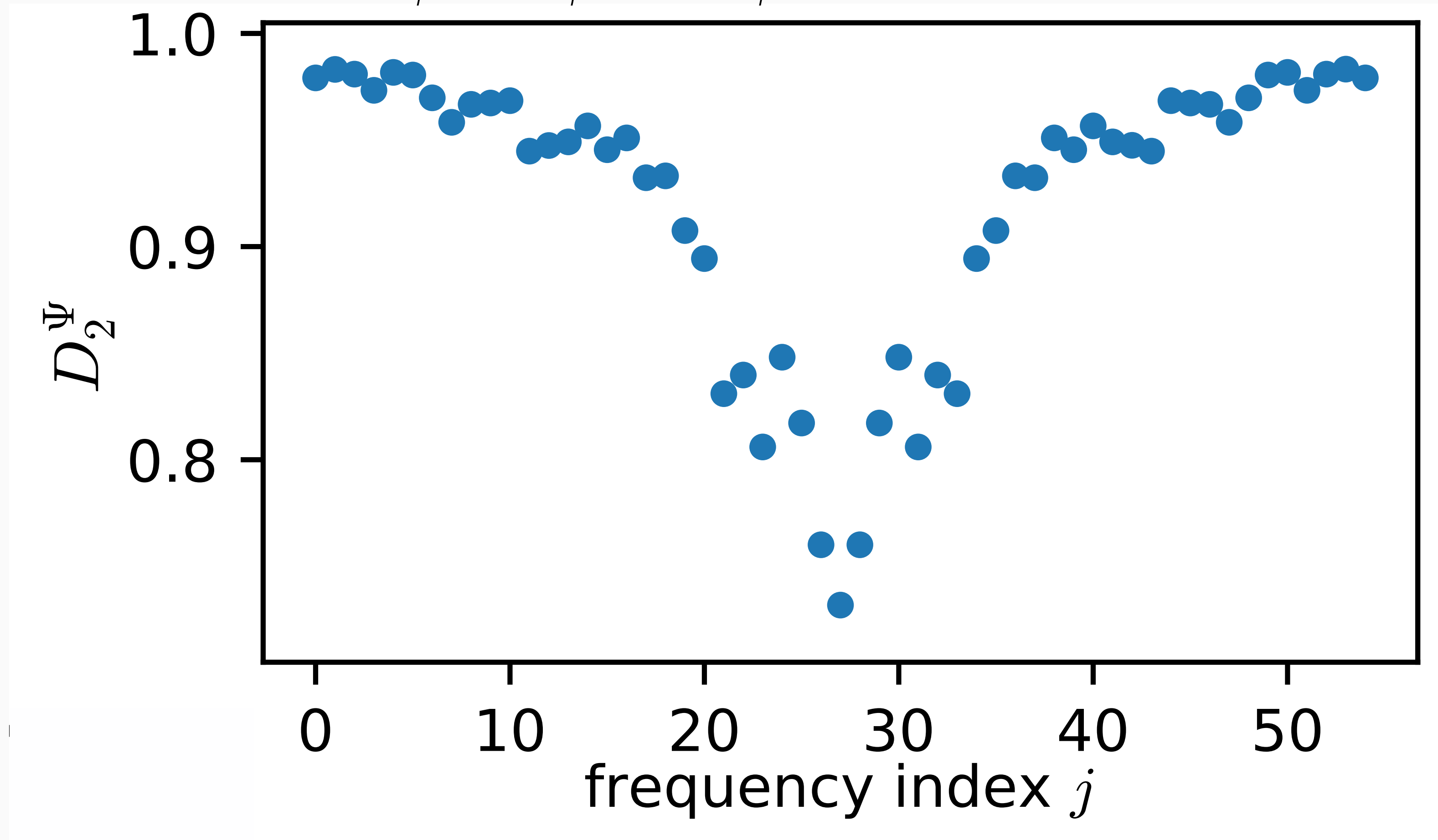
Fractal Dimensions of the Wavefunctions

Fractal Dimension D_q^ψ of state j as a function of q

$$\chi_q^{(n)}(j) = \sum_i |\psi_j^{(n)}(i)|^{2q} \underset{n\to\infty}{\sim} F_n^{-(q-1)D_q^\psi(j)}$$



q=2: participation ratio



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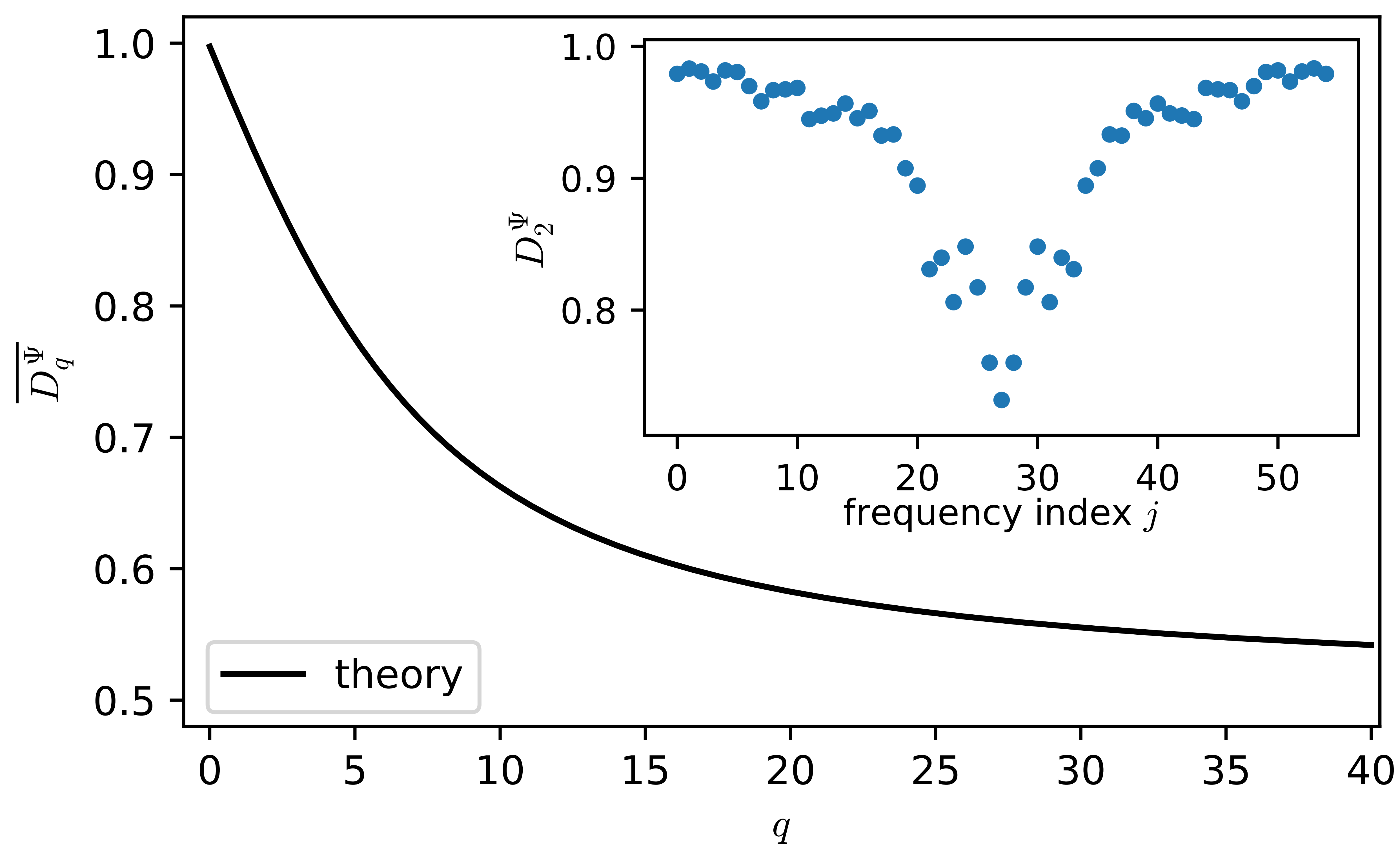
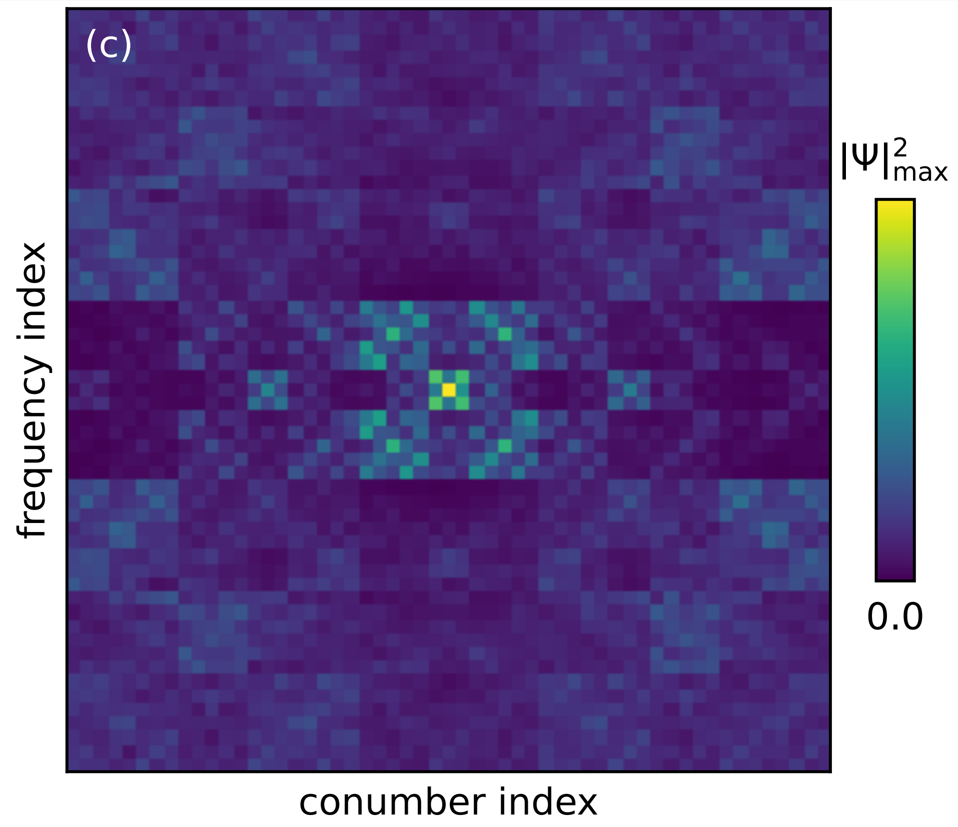
Averaged Fractal Dimension $\overline{D_q^{\psi}}$

$$\langle \chi_q^n(j) \rangle_j = \frac{1}{F_n} \sum_j \chi_q^{(n)}(j) \underset{n\to\infty}{\sim} F_n^{-(q-1)\overline{D_q^{\psi}}}$$





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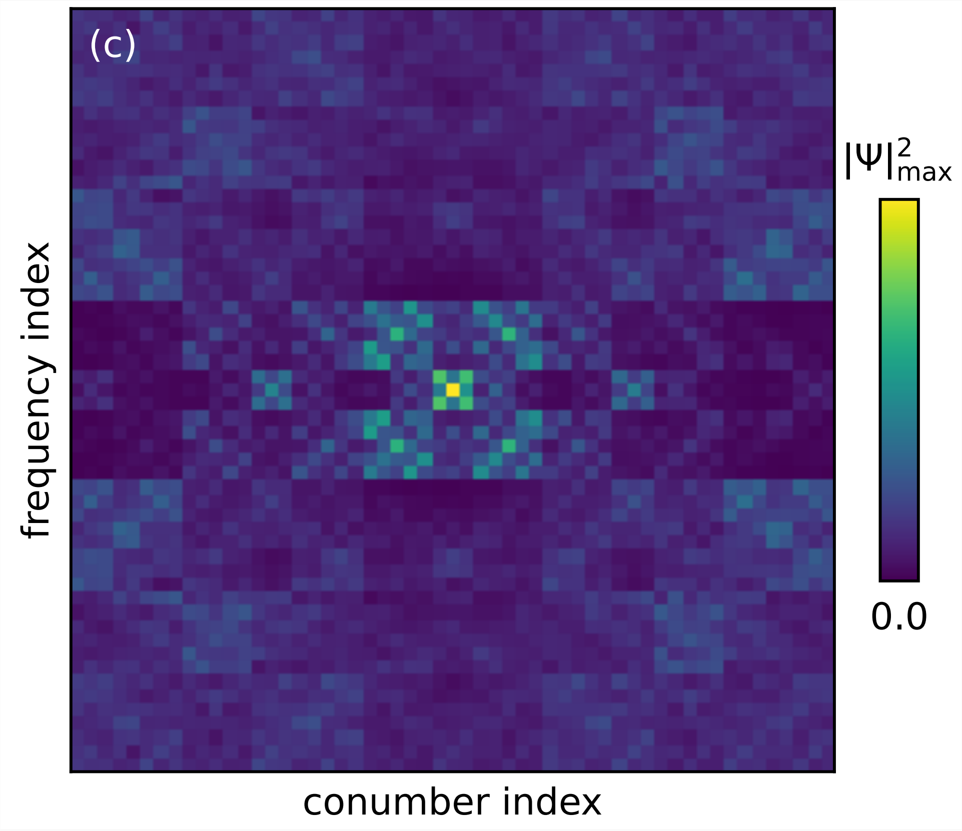

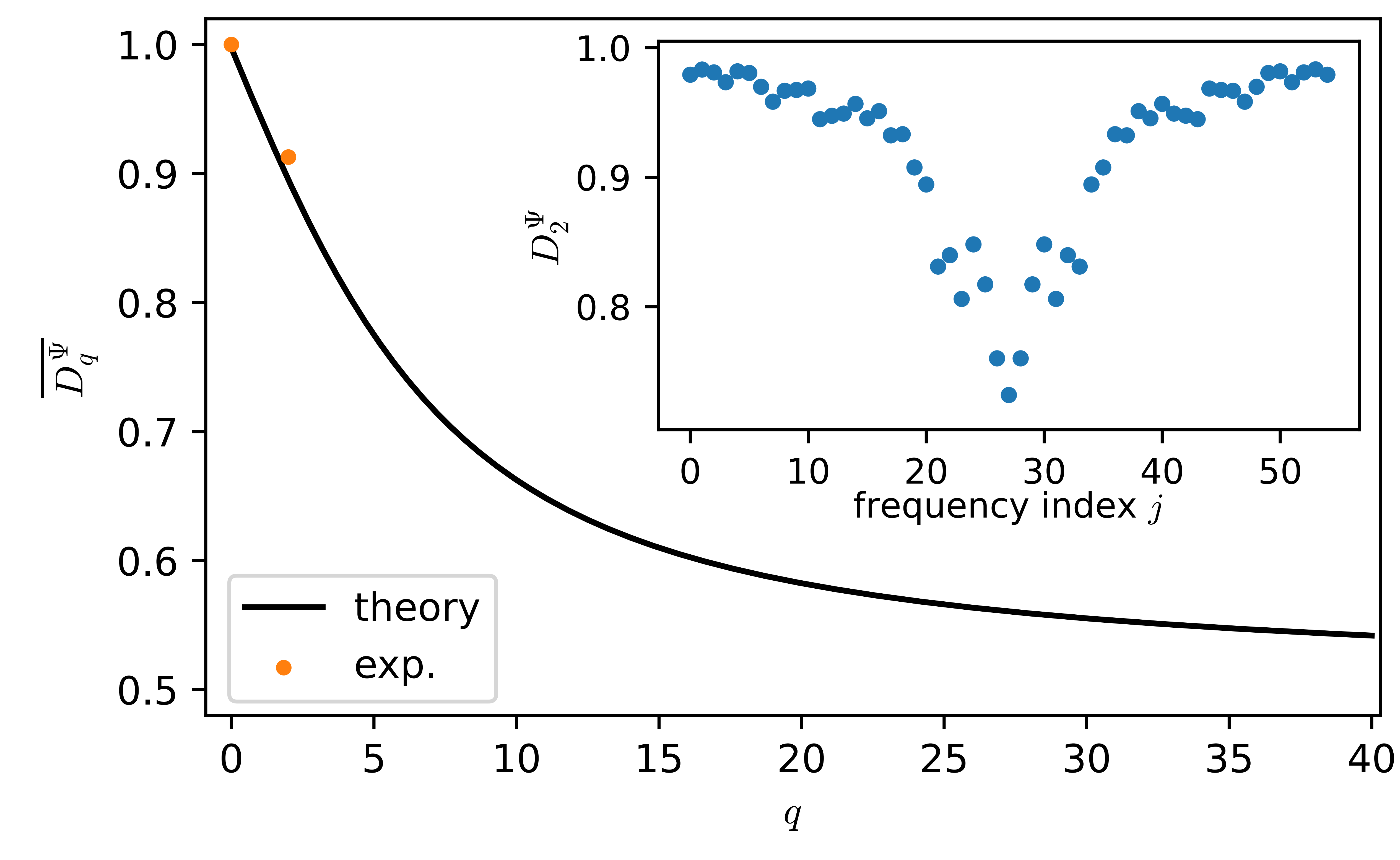

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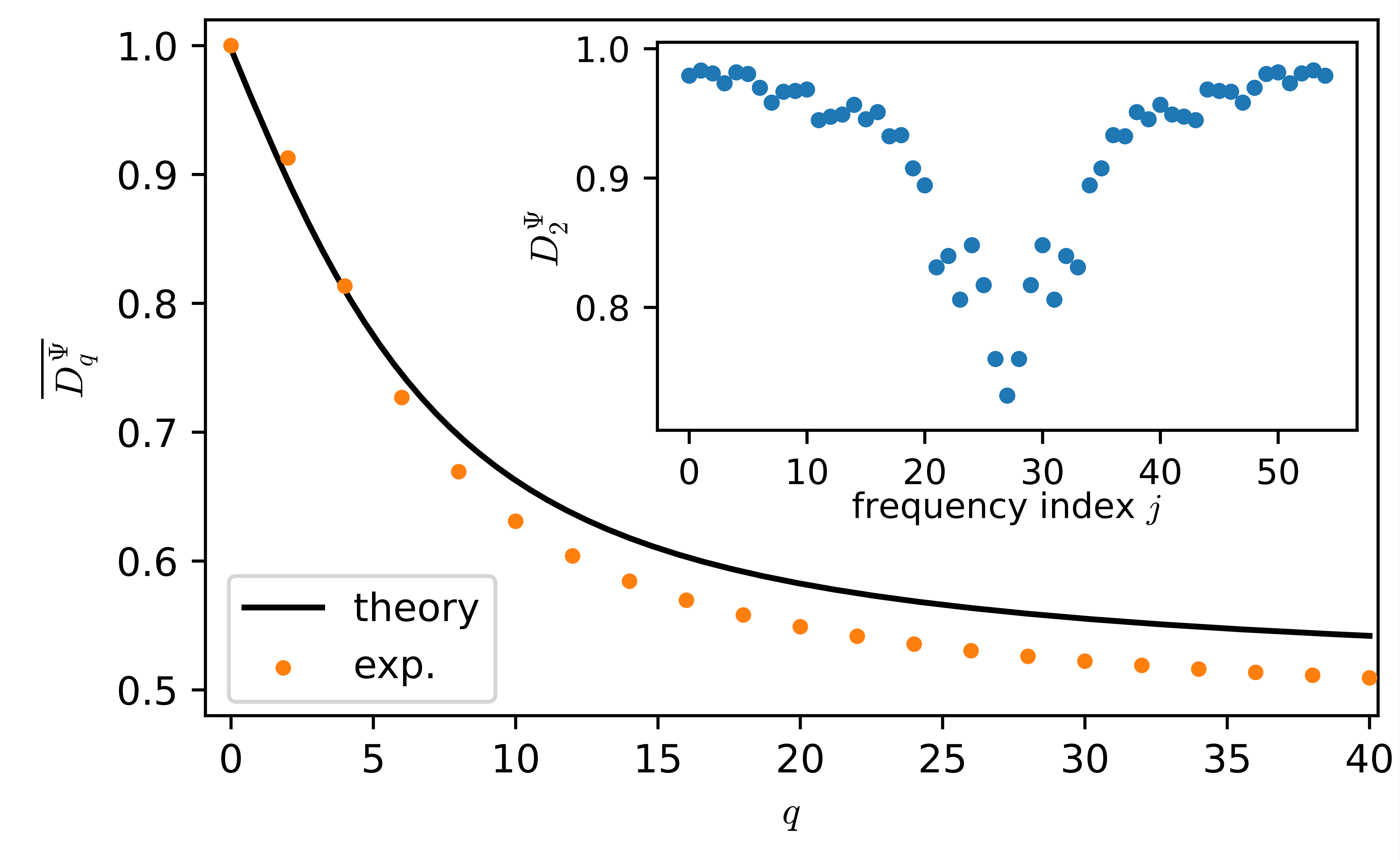

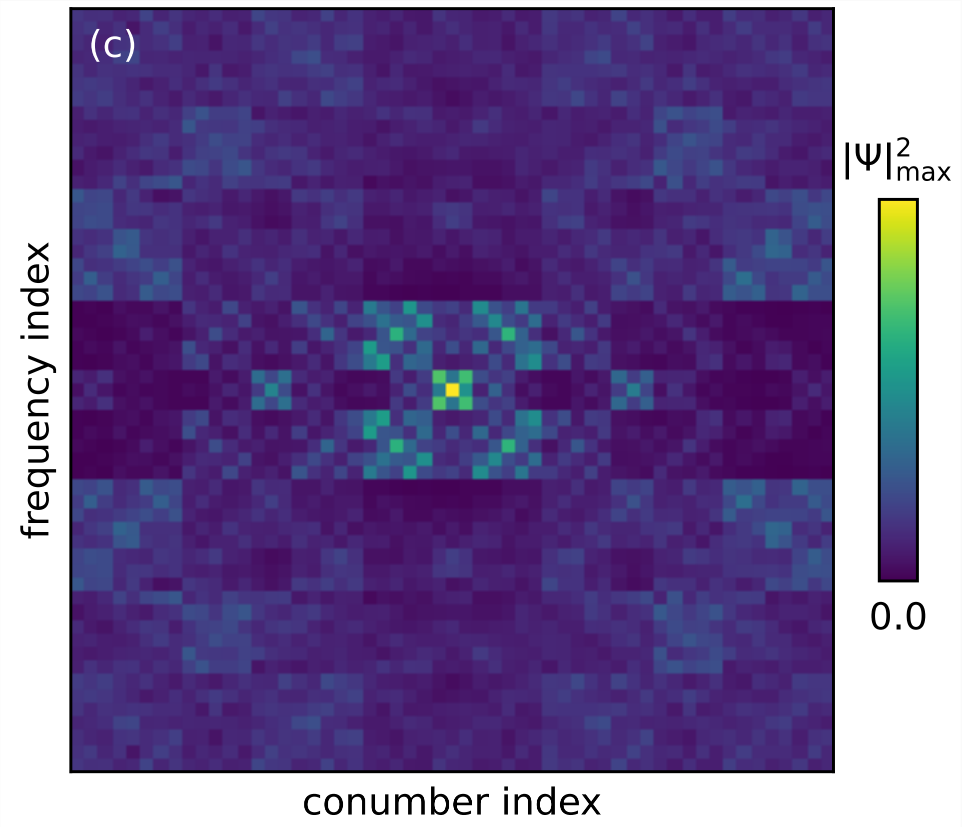

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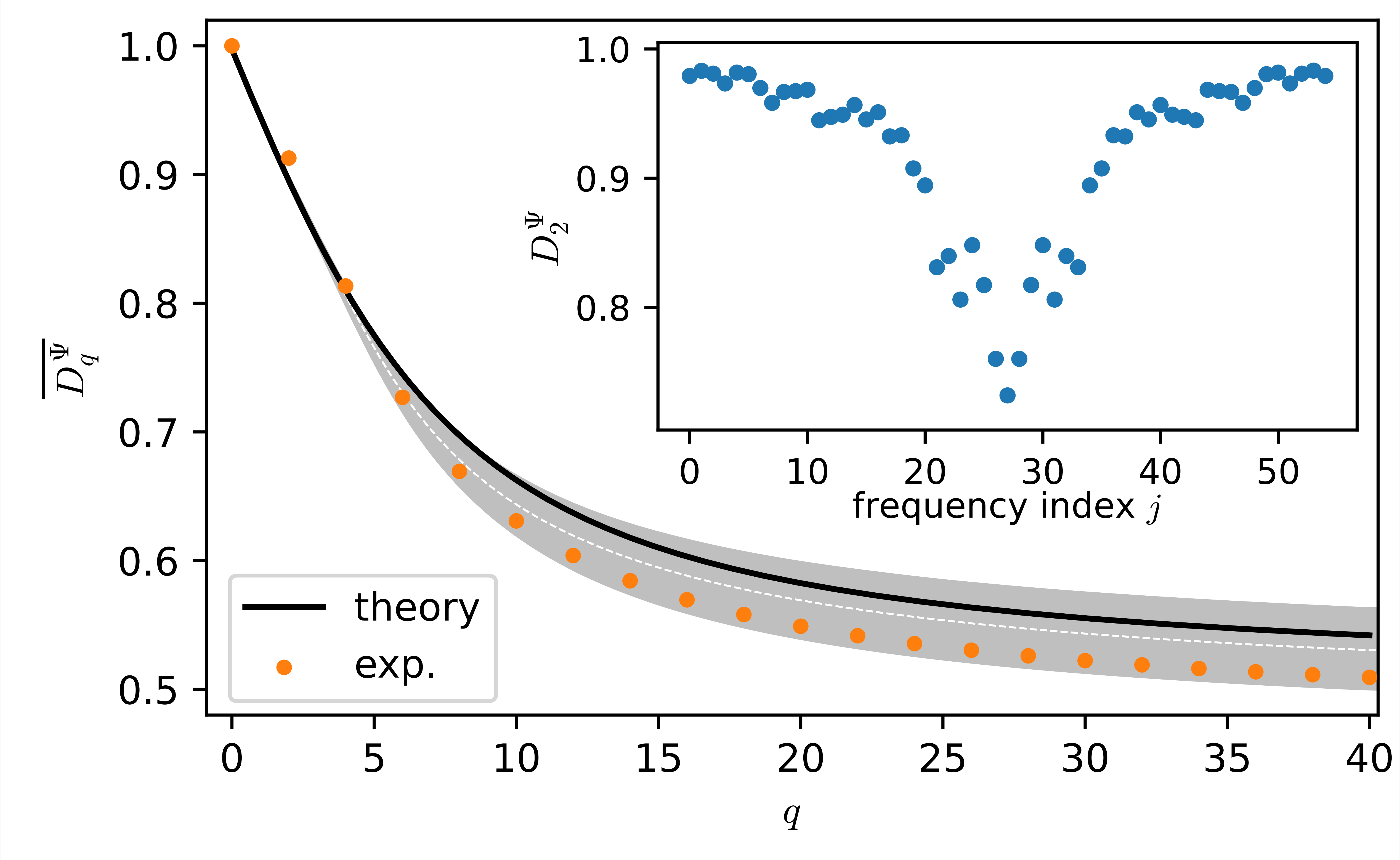
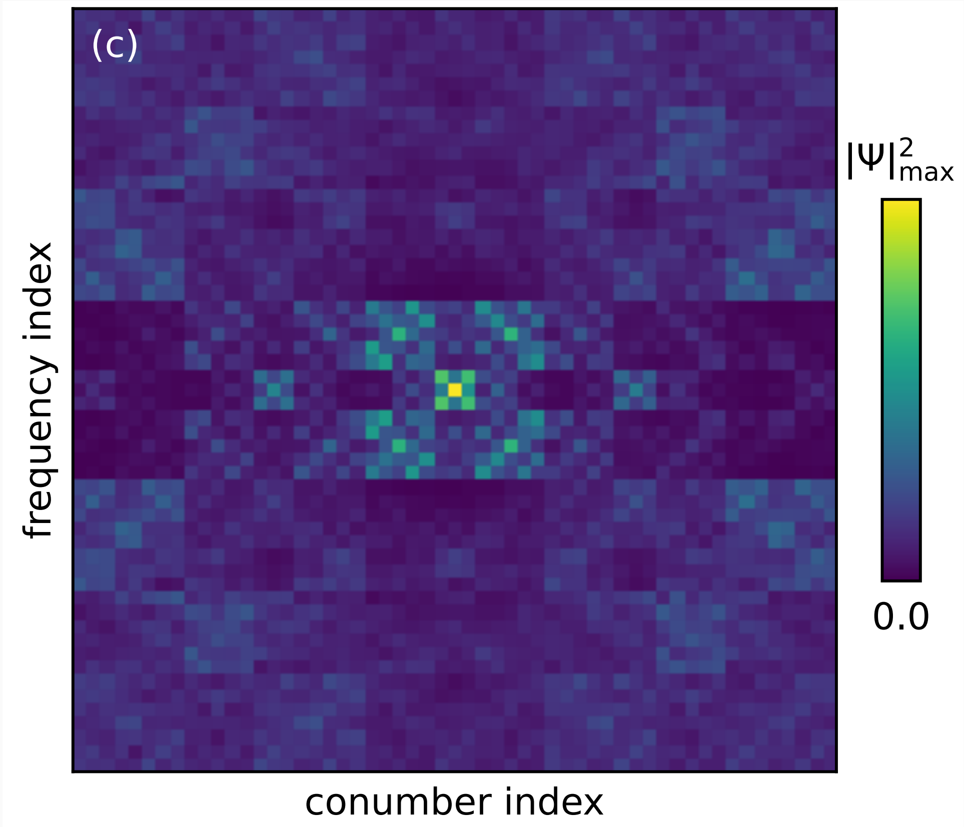




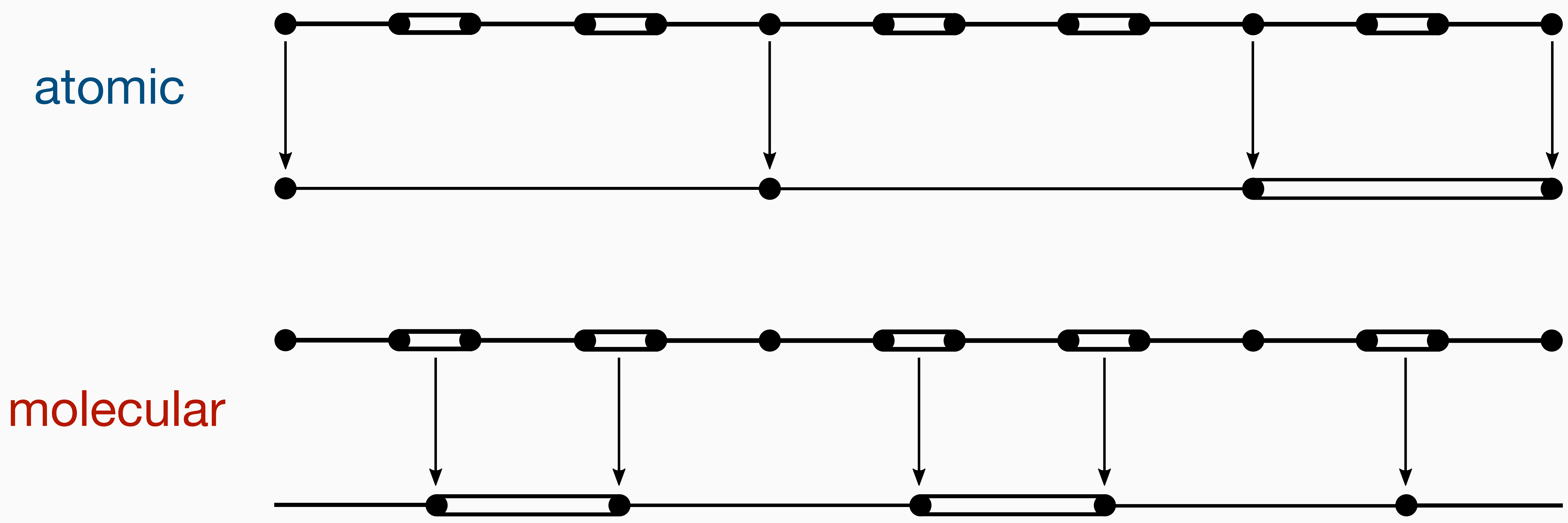
Reasons for the offset:

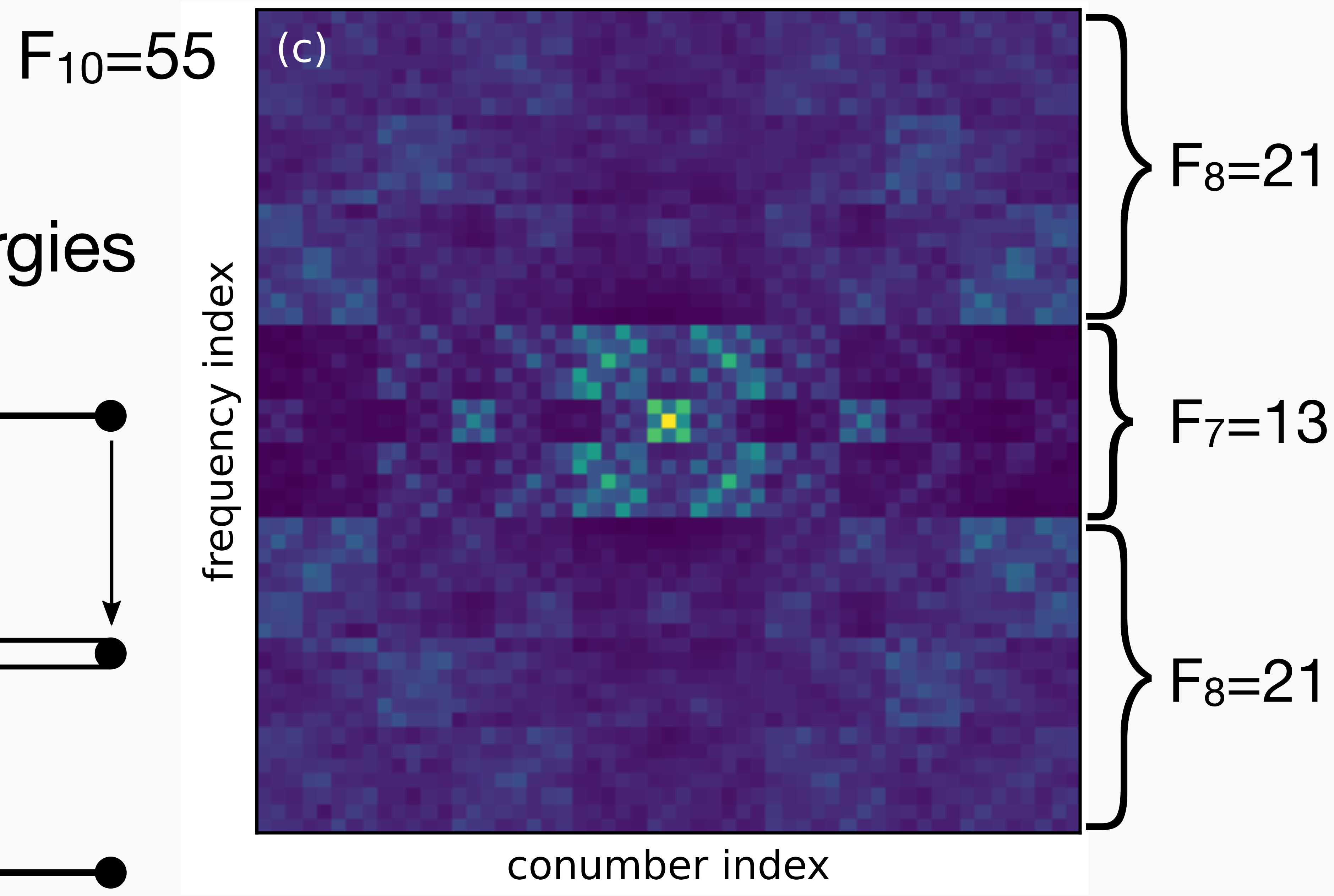
- Finite system size
- Experimental fluctuations

For more details see arXiv:2207.13755

Recursive Construction

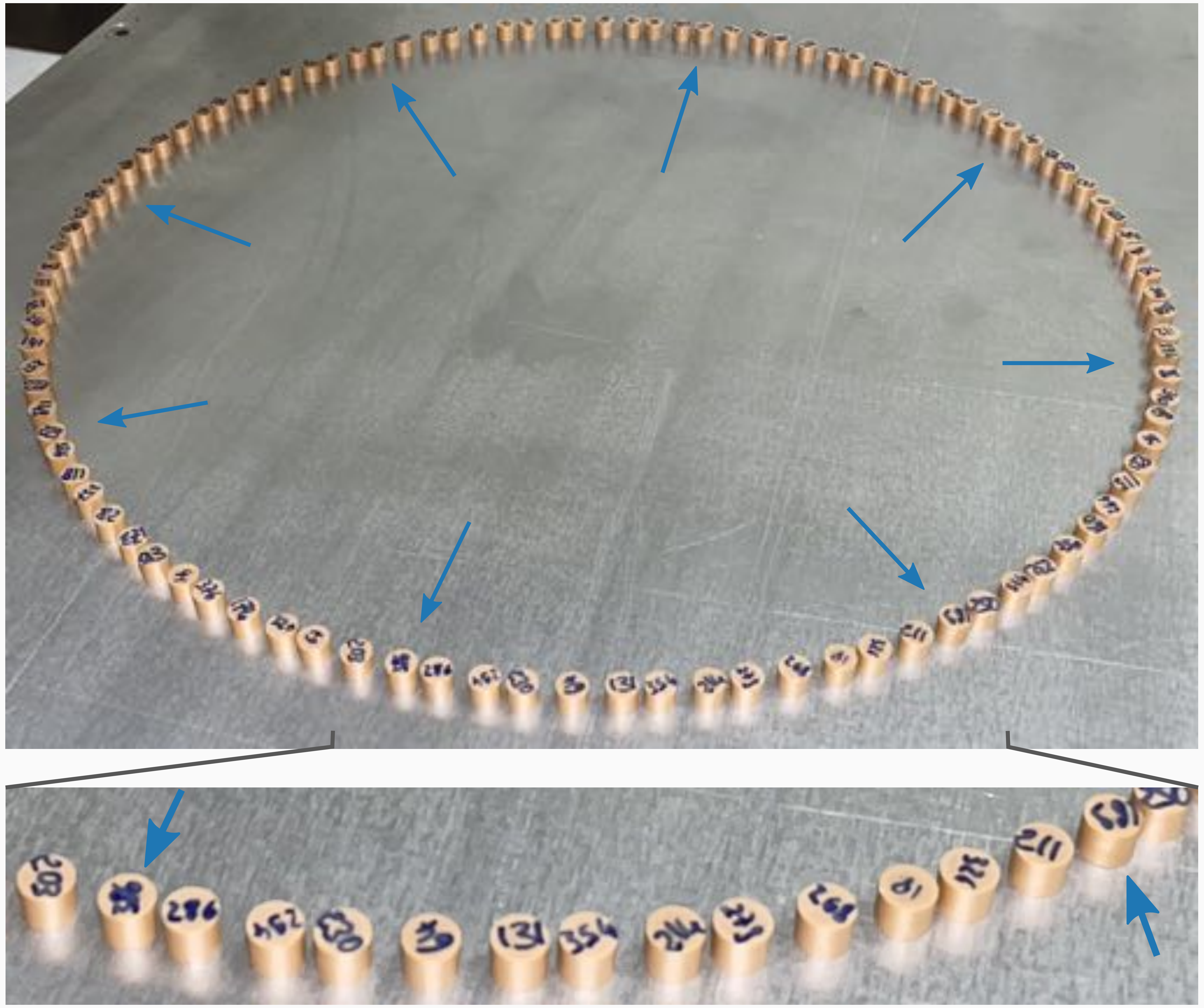
- renormalization theory to describe recursive construction
- different recursive construction for atomic and molecular sites/energies





- for atomic sites/energies: $\psi^{(n)} = \tilde{\lambda} \cdot \psi^{(n-3)}$
- for molecular sites/energies: $\psi^{(n)} = \lambda \cdot \psi^{(n-2)}$

Study of the Recursive Construction with Circular Chains



Different method to experimentally realise the periodic approximates of infinite length

- small motif length —> few permutations
- basic motif ($F_7 = 13$) repeated 8 times
- ring of ≈100 resonators
- no edge states
- frequency-bands get populated
- average over periodic sites

Study of the Recursive Construction with Circular Chains

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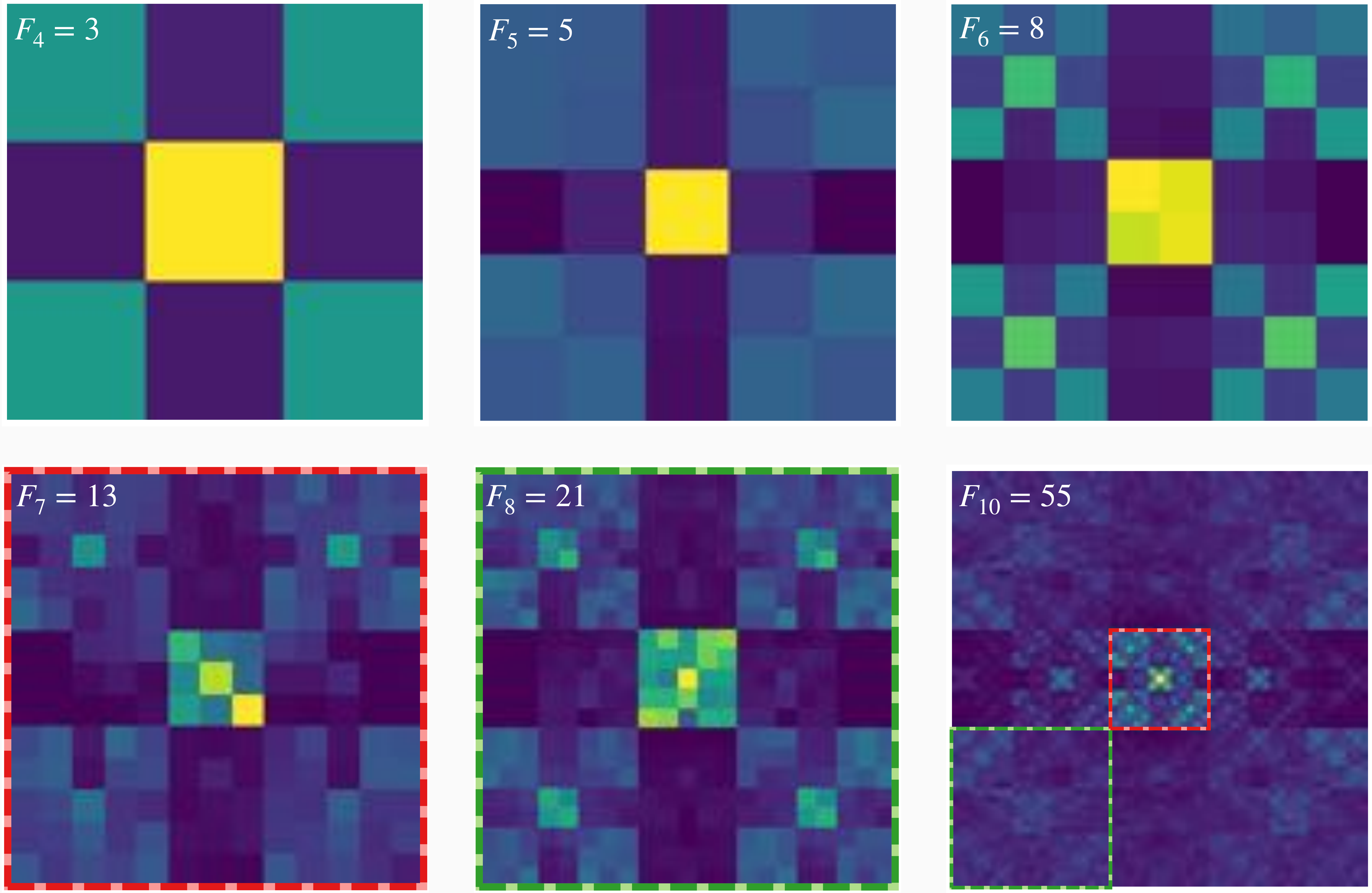
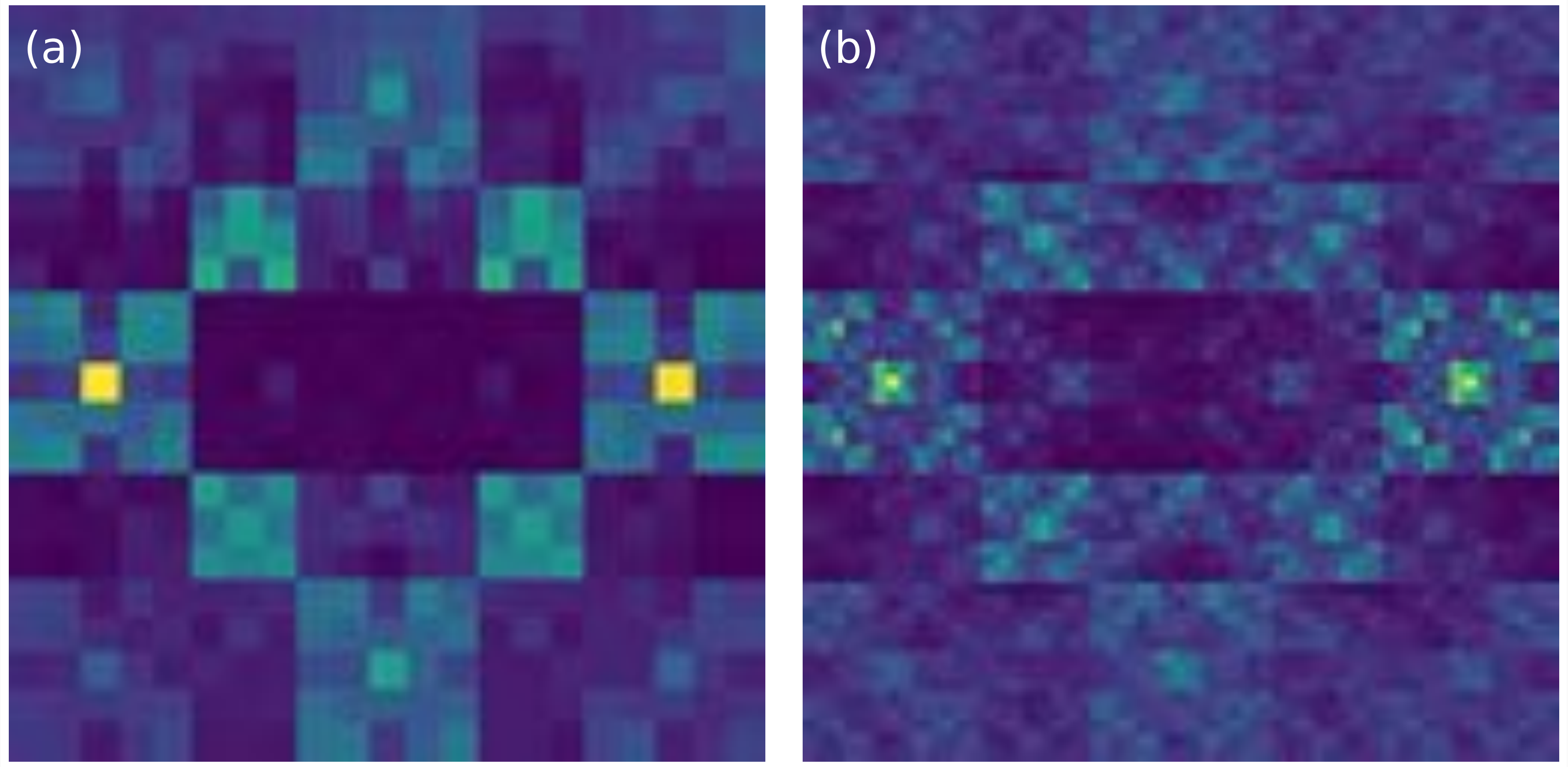


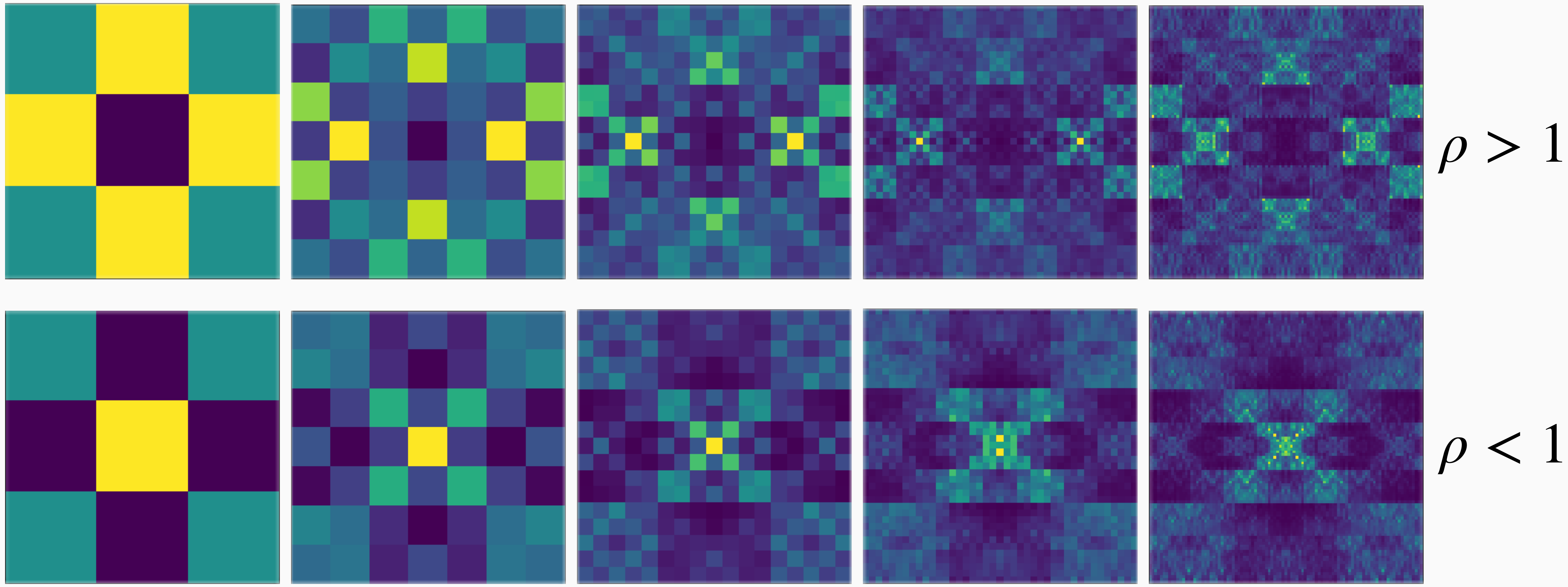
Outlook

- Interchange weak and strong couplings: $t_A \rightleftarrows t_B \quad \rightarrow \quad \rho > 1$



- other metallic means: silver mean, bronze mean, ….

$$-> F_{n+1} = m \cdot F_n + F_{n-1}$$



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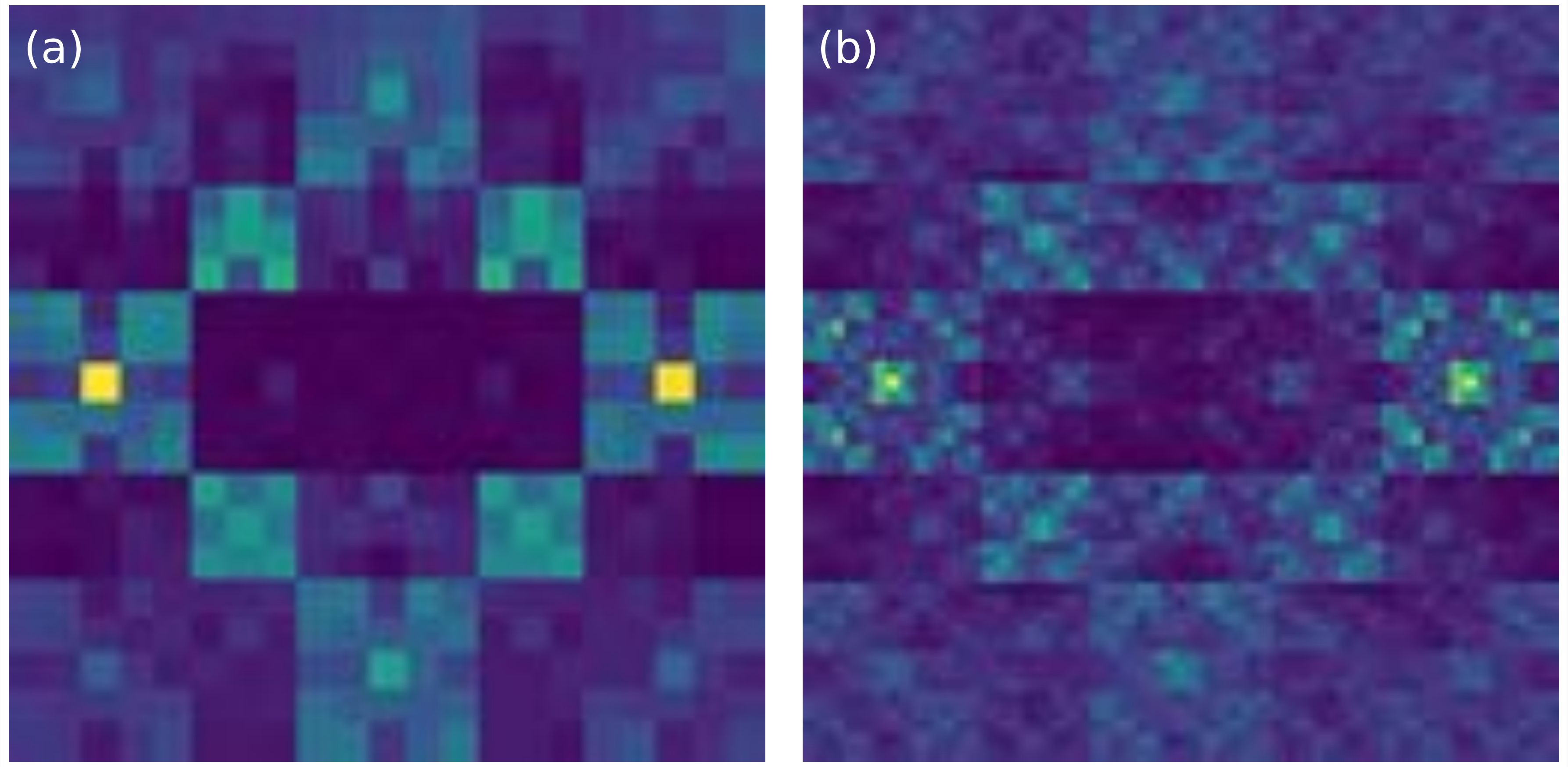

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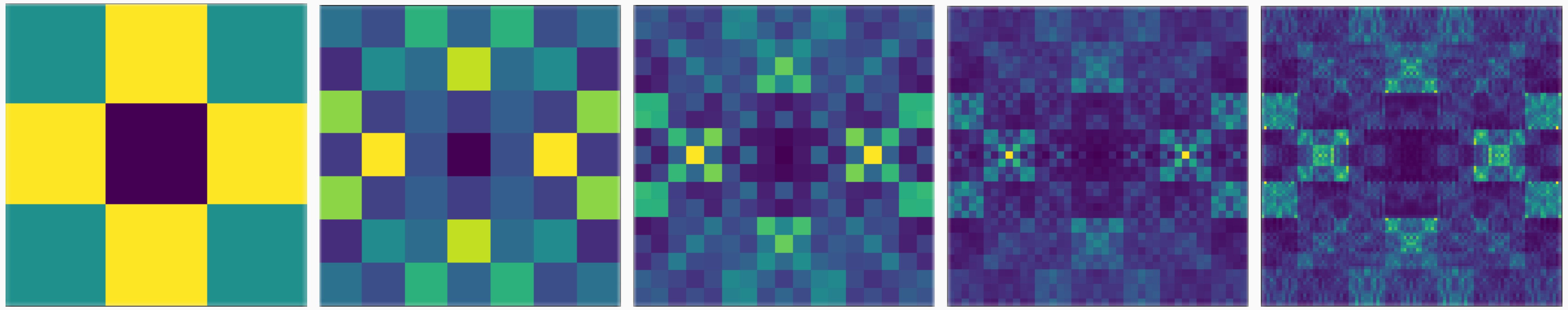

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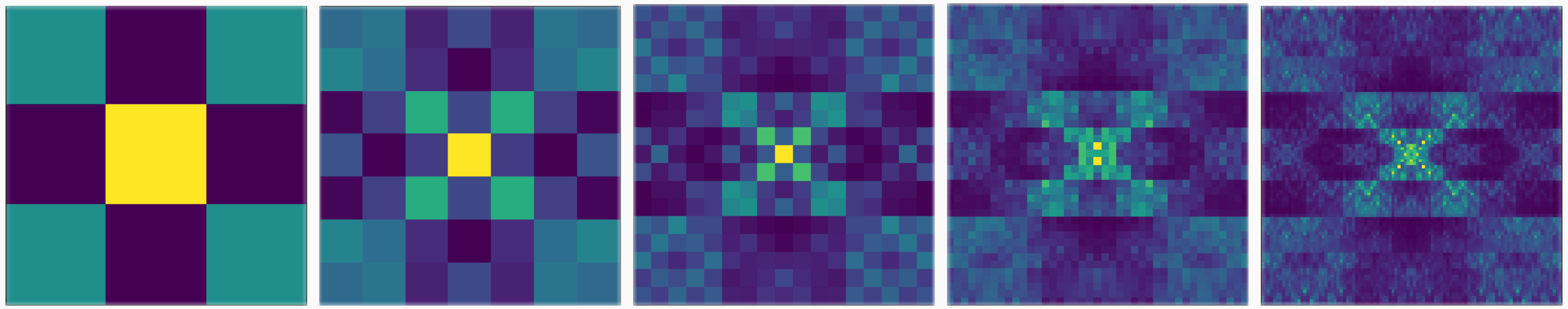

$\rho < 1$

Thank you for your attention!