Critical velocity for superfluidity in reduced dimensions

From matter to light quantum fluids

Juliette Huynh

December 8, 2022

JH, Mathias Albert, and Pierre-Élie Larré. (2022), Phys. Rev. A 105, 023305

STLight
Quantum-SOPHA

The project

Critical velocity for superfluidify in 1D & 2D

JH

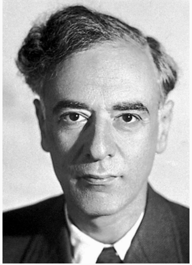
Frédéric Hébert

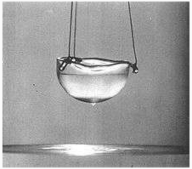
Pierre-Élie Larré

Mathias Albert

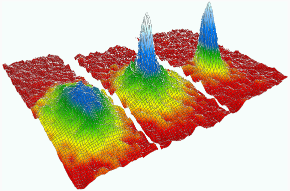
Theoretical Physics team @ Inphyni Nice

Introduction





1st evidence of BEC

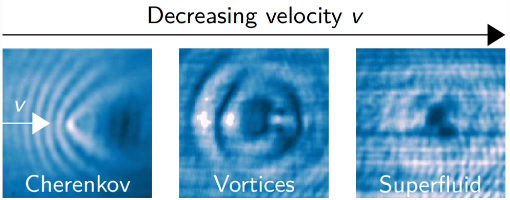


JILA, MIT 1995

SF observed in many systems:

- **BEC** (Ketterle, Cornell, Dalibard...)
- **Polaritons condensates** (Richard, Bramati, Bloch, Amo...)
- **Paraxial fluids of light** (Fleischer, Faccio, Michel, Bellec, Glorieux...)

CRITICAL VELOCITY FOR SUPERFLUIDITY



Michel, Nat. Commun (2018)
Eloy, EPL (2021)

- Flow past an impurity without any loss of kinetic energy when $v < v_c$
- Landau criterion: small obstacles only

State of the art

1D: wide or narrow gaussian, cubic NL (Hakim, Leboeuf, Pavloff)

2D: wide impenetrable cylinder, cubic NL (Frisch, Rica, Swerger, Berloff, Dalibard)

Our contribution

- Arbitrary obstacle
- General nonlinearity $g(n)$
- Treatment of the losses (1D)
- Analytical results in 1D & 2D

Dilute ultracold atomic gases $(n = |\psi|^2)$

$$i\hbar\partial_t\psi = \left[-\frac{\hbar^2}{2m}\nabla^2 + U(\boldsymbol{r}) + gn^\nu\right]\psi$$

Paraxial fluids of light $(I = |E|^2)$

$$i\partial_z E = \left[-\frac{\nabla^2}{2k} + U(\boldsymbol{r}) - \frac{\omega n_2}{c}\frac{I}{1+I/I_{\text{sat}}} - \frac{i\alpha}{2}\right]E$$

Our contribution

Arbitrary obstacle

General nonlinearity $g(n)$

Treatment of the losses (1D)

Analytical results in 1D & 2D

Dilute ultracold atomic gases $(n = |\psi|^2)$

$$i\hbar\partial_t\psi = \left[-\frac{\hbar^2}{2m}\nabla^2 + U(\boldsymbol{r}) + gn^\nu\right]\psi$$

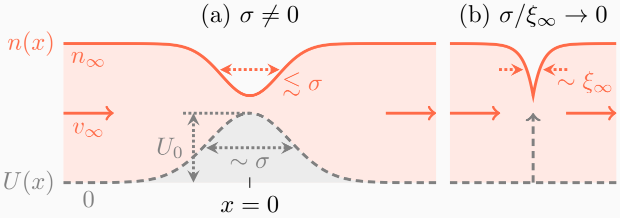
Paraxial fluids of light $(I = |E|^2)$

$$i\partial_z E = \left[-\frac{\nabla^2}{2k} + U(\boldsymbol{r}) - \frac{\omega n_2}{c}\frac{I}{1 + I/I_{\mathrm{sat}}} - \frac{i\alpha}{2}\right]E$$

Superfluid hydrodynamics

1D mean-field regime, generalized nonlinear Schrödinger equation (Gross-Pitaevskii)

$$i\partial_t \psi = \left[-\frac{1}{2}\partial_{xx} + U(x) + g(|\psi|^2) - \frac{i\gamma}{2} \right] \psi \tag{1}$$



Natural units

$x \to \xi_\infty$

$v \to c_\infty$

$E \to \mu_\infty$

Madelung transformation: hydrodynamic picture $\psi = \sqrt{n}\exp\{i\int dx\, v\}$

$$\partial_t v = -\partial_x\left(\frac{v^2}{2} + U(x) + g(n) - \frac{1}{2}\frac{\partial_{xx}\sqrt{n}}{\sqrt{n}}\right)$$

$$\partial_t n + \partial_x(nv) = -\gamma n$$

$$\partial_t v = -\partial_x \left(\frac{v^2}{2} + U(x) + g(n) - \frac{1}{2} \frac{\partial_{xx}\sqrt{n}}{\sqrt{n}} \right)$$

$$\partial_t n + \partial_x (nv) = \cancel{-\gamma n}$$

Without losses

Stationary solutions

Far away from the obstacle:

- density $n \to n_\infty$
- velocity $v \to v_\infty$

1D MODEL

$$\cancel{\partial_t v} = -\partial_x \left(\frac{v^2}{2} + U(x) + g(n) - \frac{1}{2} \frac{\partial_{xx}\sqrt{n}}{\sqrt{n}} \right)$$

$$\cancel{\partial_t n} + \partial_x (nv) = \cancel{-\gamma n}$$

Without losses

Stationary solutions

Far away from the obstacle:

- density $n \to n_\infty$
- velocity $v \to v_\infty$

1D MODEL

$$\cancel{\partial_t v} = -\partial_x \underbrace{\left(\frac{v^2}{2} + U(x) + g(n) - \frac{1}{2}\frac{\partial_{xx}\sqrt{n}}{\sqrt{n}}\right)}_{= \mathbf{C^{te}} = \frac{v_\infty^2}{2} + g(n_\infty)}$$

$$\cancel{\partial_t n} + \partial_x \underbrace{(nv)}_{= \mathbf{C^{te}} = n_\infty v_\infty} = \cancel{-\gamma n}$$

Without losses

Stationary solutions

Far away from the obstacle:

- density $n \to n_\infty$
- velocity $v \to v_\infty$

Analytical results

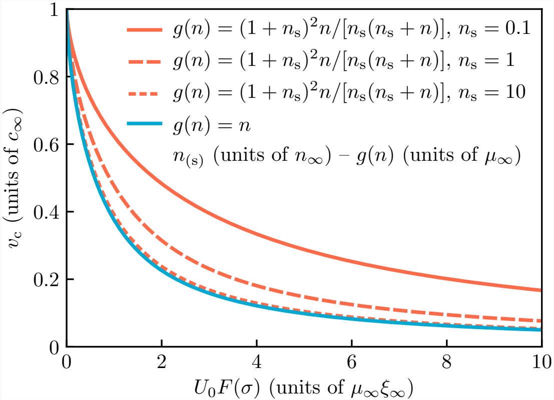
Narrow obstacle

Small σ: $U(x) \to U_0 F(\sigma)\delta(x)$

$$U_0 F(\sigma) = \sqrt{2}\left[-\frac{v_c^2}{2}\left(1 - \frac{1}{n_{0,c}}\right)^2 - g(1) + \frac{g(1) - G(1) + G(n_{0,c})}{n_{0,c}}\right]^{\frac{1}{2}}, \quad (6a)$$

$$\frac{n_{0,c}}{1 - n_{0,c}}\left(g(1) - G(1) - g(n_{0,c})n_{0,c} + G(n_{0,c})\right) = v_c^2. \quad (6b)$$

With $G(n) = \int g(n)dn$ and $n_{0,c} = n(x = 0, v_\infty = v_c)$.



Nonlinearity

Saturable: $g(n) = \frac{(1+n_s)^2}{n_s} \frac{n}{n+n_s}$

Cubic: $g(n) = n$

- Landau criterion OK for $U_0 F(\sigma) \to 0$
- $v_c \searrow$ as $U_0 F(\sigma) \nearrow$
- Light SF more robust

Wide obstacle

Large σ: $U(x) = U_0 f\left(|x|/\sigma\right) \approx U_0 \left[1 + f''(0)\frac{x^2}{2\sigma^2}\right]$ → Multiple-scale analysis with gradients of $n(x)$ in powers of $1/\sigma \ll 1$. Hakim, Phys. Rev. E (1997)

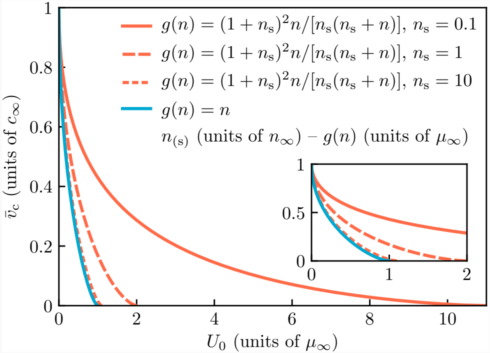
To higher order in $1/\sigma$, taking into account the curvature of the obstacle:

$$v_c = \bar{v}_c + \frac{1.466}{2^{\frac{5}{3}}} \frac{\left[U_0 |f''(0)| \bar{n}_{0,c}\right]^{\frac{2}{3}}}{\bar{v}_c \left(\frac{1}{\bar{n}_{0,c}} - \bar{n}_{0,c}\right) \left[\frac{3\bar{v}_c^2}{\bar{n}_{0,c}^3} + g''(\bar{n}_{0,c})\bar{n}_{0,c}\right]^{\frac{1}{3}}} \frac{1}{\sigma^{\frac{4}{3}}}. \tag{7}$$

- Positive correction to $\bar{v}_c$: **SF favoured**
- $\sigma^{-\frac{4}{3}}$-dependence no matter $g(n)$ and $U(x)$: universal behaviour

Wide obstacle

$\bar{v}_c$ obtained at $\mathcal{O}(1/\sigma^0)$

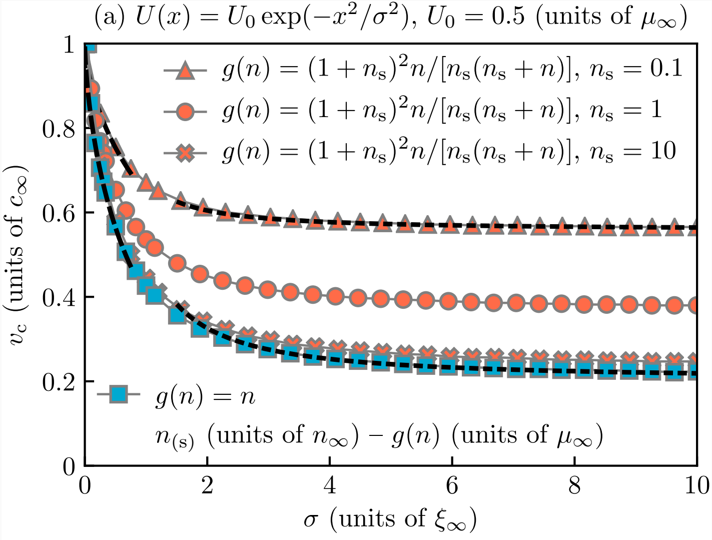


- Same remarks as for the narrow obstacle
- $\bar{v}_c = 0$, $n_{0,c} = 0$ for $U_0 \geq U_{0,\max}$: fluid separated in 2 $\rightarrow$ **no SF**

In reality: we cannot neglect the density variations and the hydraulic approximation fails $\rightarrow$ Josephson treatment with WKB approach $v_c = A \exp\left\{-\sqrt{2}\int_{-x_{\rm cl}}^{x_{\rm cl}} dx [U(x)-\mu]^{\frac{1}{2}}\right\}$.

Numerical results

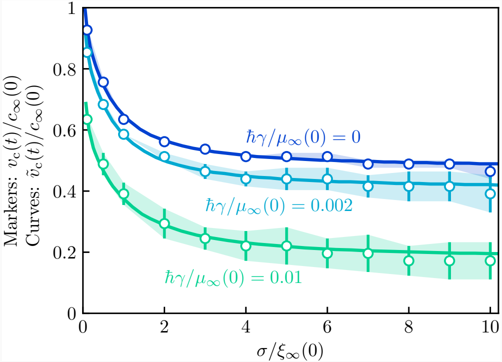
Repulsive obstacle of arbitrary width



Effect of losses

- In specific systems like SF of light, **losses** are not negligible.
- But if they manifest on a large time scale $1/\gamma$: As if the fluid were in an equilibrium state at each time t by **adiabatically** following the variations of its unperturbed density $n_\infty(t) = n_\infty(0)\exp(-\gamma t)$.

Effect of losses

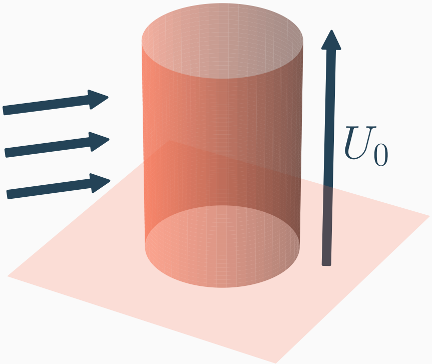


- $\tilde{v}_c(t)$: Adiabatic-evolution approximation + Relaxation algorithm
- $v_c(t)$: Brute-force numerical integration of NLS with a moving obstacle

2D

CRITICAL VELOCITY FOR SUPERFLUIDITY IN 2D

$$\frac{v^2}{2} + U(\boldsymbol{r}) + g(n) - \frac{1}{2}\frac{\nabla^2\sqrt{n}}{\sqrt{n}} = \mathrm{Cte} \quad \text{and} \quad \boldsymbol{\nabla}\cdot(n\boldsymbol{v}) = 0 \quad \rightarrow$$ **nonintegrable**



Our contribution

State of the art	Here
• $R \gg \xi_\infty$	• $R \gg \xi_\infty$
• $U_0 = +\infty$	• Arbitrary U_0
• Cubic NL	• Arbitrary NL

Frisch, Rica, Zwerger, Berloff, Dalibard

The condition to obtain the critical velocity

$R \gg \xi_\infty$:

$$\frac{v^2}{2} + U(\boldsymbol{r}) + g(n) - \frac{1}{2}\cancel{\frac{\nabla^2\sqrt{n}}{\sqrt{n}}} = \mathrm{Cte}$$

$$\downarrow$$

$$g(n) = g\left(n(v)\right) = \mathrm{Cte} - U(\boldsymbol{r}) - \frac{v^2}{2}$$

n only depends on the modulus of v

The condition to obtain the critical velocity

$\nabla \cdot (n(v)\boldsymbol{v}) = 0 \quad \rightarrow \quad$ Make that **highly nonlinear** equation **linear**

Hodograph plane: Legendre transformation $\Phi(\boldsymbol{v}) = \frac{m\boldsymbol{v}}{\hbar} \cdot \boldsymbol{r} - \phi(\boldsymbol{r})$

$$nv\frac{\partial^2 \Phi}{\partial v^2} + \frac{d(nv)}{dv}\frac{\partial \Phi}{\partial v} + v\frac{d(nv)}{dv}\frac{\partial^2 \Phi}{\partial \theta^2} = 0 \tag{8}$$

Elliptic solutions: complex-valued characteristic equation of (8)

$$\boxed{\frac{d(nv)}{dv} > 0} \tag{9}$$

Condition relating to the maximum velocity v_{max}

Treatment of the compressibility

Solve $\nabla \cdot (n\boldsymbol{v}) = 0$ to find v_{max}

First approximation: $n = \text{Cte} \rightarrow$ **Incompressible fluid** ✗

Solution: include the **compressibility** $\chi \propto \frac{1}{c_\infty^2}$ with a perturbative Janzen-Rayleigh expansion of the potential velocity

$$\phi = \phi_0 + \chi\phi_1 + \chi^2\phi_2 + \ldots \text{ with } \boldsymbol{v} = \frac{\hbar}{m}\nabla\phi$$

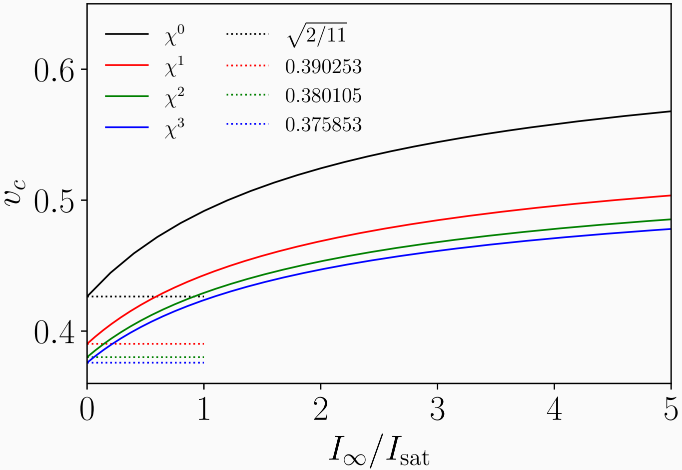
Critical velocity for superfluidity in 2D

We want $v_c(U_0)$:

2 DIFFERENT CASES

Impenetrable obstacle	Penetrable obstacle
$U_0 \gg \mu_\infty$	$U_0 < \mu_\infty$
The fluid goes around the obstacle	The fluid goes around or crosses the obstacle
e.g. Rica, Phys. D (2001)	Theory for this?

The impenetrable obstacle: $U_0 \gg \mu_\infty$

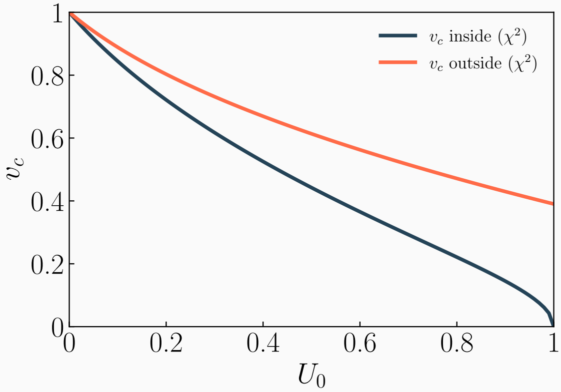


- v_c indep. of U_0
- **Superfluids of light**
  $g(I) = \frac{\alpha I}{1+I/I_{\text{sat}}}$
- $I_\infty/I_{\text{sat}} = 0$
  Cubic NL $\to$ Rica

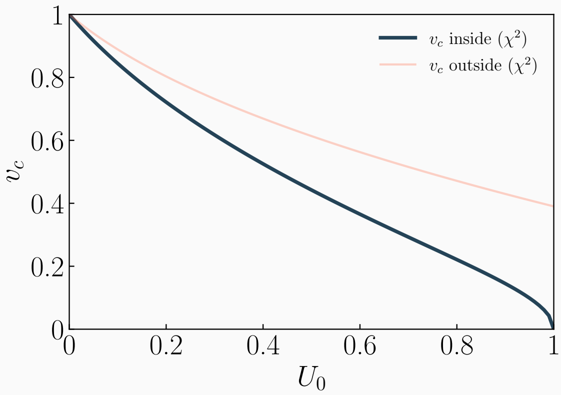
The penetrable obstacle: $U_0 < \mu_\infty$

v_c does depend on U_0

2 possible breakdowns of SF $\rightarrow$ **2 velocities**

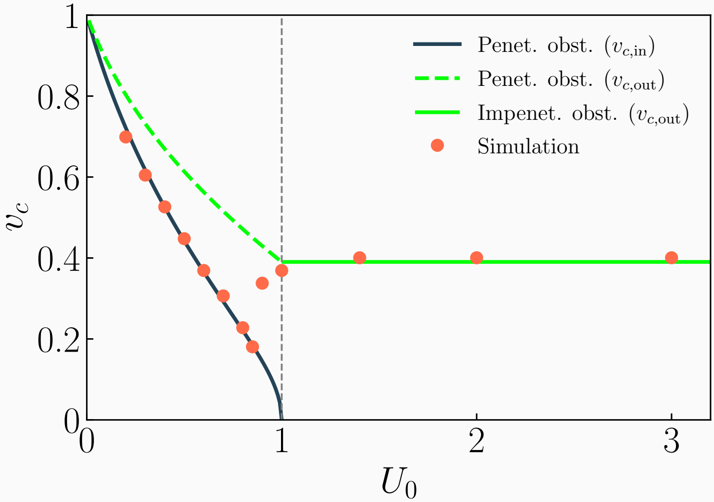


THE PENETRABLE OBSTACLE: $U_0 < \mu_\infty$



The **real** critical velocity for SF is the **lowest** of the 2

Comparison analytical/numerical results



Similar results in Shin's group (exp. BEC)
2015, 2022

Conclusion

Conclusion

- 1D & 2D critical velocity beyond the Landau criterion

  - **General local nonlinearity** & **arbitrary obstacle**
  - Treatment of **particle losses** (1D)
  - **Analytical results** in the mean-field regime

- Method: search for the existence of stationary unperturbed solutions conditioned by the velocity
- Perspectives: How do **quantum fluctuations** affect the mean-field SF transition? And what about **disordered systems** ?

JH, Mathias Albert, and Pierre-Élie Larré. (2022), Phys. Rev. A 105, 023305

THANK YOU FOR YOUR ATTENTION!