Wigner-Smith time delay matrix in multichannel disordered wires

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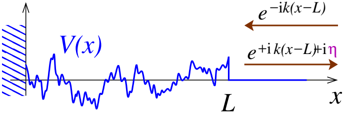
december 7, 2022

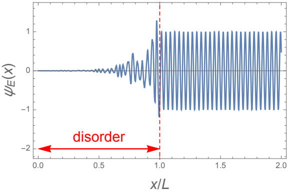
Stationary scattering state :

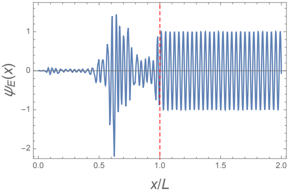
$$-\psi_\varepsilon''(x) + V(x)\psi_\varepsilon(x) = \varepsilon\,\psi_\varepsilon(x)$$



$$\psi_\varepsilon(x) \underset{x\geq L}{=} \frac{1}{\sqrt{2\pi v_\varepsilon}}\left(e^{-\mathrm{i}k(x-L)} + e^{+\mathrm{i}k(x-L)+\mathrm{i}\,\eta(\varepsilon)}\right)$$

with $v_\varepsilon = \partial\varepsilon/\partial k = 2k$



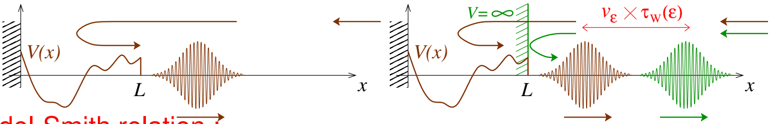


resonant state

Time spent in the disordered region: $\tau(\varepsilon) \sim \hbar \int_0^L \mathrm{d}x\,|\psi_\varepsilon(x)|^2$

Phase shift and time delay

- scattering phase shift : $\eta(\varepsilon)$
- Wigner time delay : $\tau_W(\varepsilon) \stackrel{\text{def}}{=} \hbar \dfrac{\partial\eta(\varepsilon)}{\partial\varepsilon}$



Friedel-Smith relation :

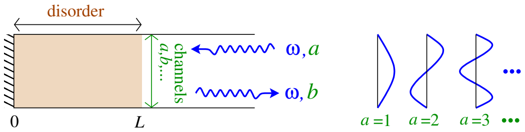
$$\int_0^L \mathrm{d}x \, \overbrace{|\psi_\varepsilon(x)|^2}^{\rho(x;E)} = \frac{1}{2\pi\hbar}\left(\tau_W(\varepsilon) + \frac{\hbar}{2\varepsilon}\sin\eta(\varepsilon)\right)$$

(for centro-sym. potentials: Friedel, Nuovo Cimento '58 ; Smith, Phys. Rev. '60)

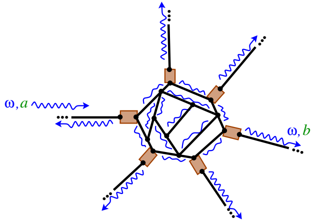
Why $\tau_W(\varepsilon)$ is an important quantity ? DoS interpretation :

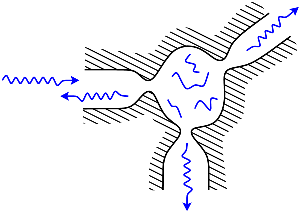
$$\rho_L(\varepsilon) = \int_0^L \mathrm{d}x \, \rho(x;E) \simeq \frac{1}{2\pi\hbar}\tau_W(\varepsilon)$$

Scattering of a wave in a multichannel disordered waveguide



Scattering in complex structures





Scatt. state in channel a for wave injected in channel b : $\psi_\varepsilon^{(b)}(x \in a)$

$N \times N$ matrix wave function $\Psi_\varepsilon(x)$ and scattering matrix $S_{ab}(\omega)$

Wigner-Smith time delay matrix :

$$\mathcal{Q} \stackrel{\text{def}}{=} -\mathrm{i}\,\mathcal{S}^\dagger \frac{\partial \mathcal{S}}{\partial \omega} \qquad \to \begin{cases} N = 1 \;\Rightarrow\; \mathcal{S} = e^{\mathrm{i}\eta} \\ \mathcal{Q} \equiv \tau_W = \hbar \frac{\partial \eta}{\partial \varepsilon} \end{cases}$$

encodes several characteristic times

Review : CT, arXiv:1507.00075v6 [Physica E**82**, 16 (2016)]

An exact relation

$$\int_{\text{scatt. region}} \mathrm{d}x\, \Psi_\varepsilon^\dagger(x)\Psi_\varepsilon(x) = \frac{1}{2\pi\hbar}\left(\mathcal{Q} + \frac{\mathcal{S}-\mathcal{S}^\dagger}{4\mathrm{i}\omega}\right) \simeq \frac{\mathcal{Q}}{2\pi\hbar}$$

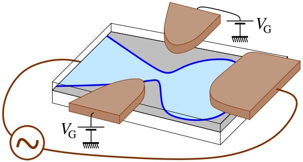
[for metric graphs: CT & Büttiker, PRB**67** (2003)]

Application for coherent electronic transport

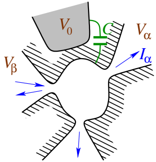
$\mathcal{Q}_{aa}(\varepsilon)$: partial DoS controls the charge injected from channel/contact a

- AC transport : $Z(\omega) = \dfrac{1}{-\mathrm{i}\omega\, C_\mu} + R_q + \mathcal{O}(\omega)$

Büttiker, Prêtre & Thomas, PRL (1993) ; PLA (1993) ; Z.Phys.B (1994)



- Nonlinear transport : $I_\alpha = \sum_\beta G_{\alpha\beta} V_\beta + \sum_{\beta,\gamma} G_{\alpha\beta\gamma} V_\beta V_\gamma + \cdots$

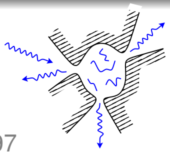


Büttiker, J.Phys.C (1993) ; Christen & Büttiker, EPL (1996)

Review : CT, arXiv:1507.00075v6 [Physica E**82**, 16 (2016)]

Wigner-Smith matrix in quantum dots

Quantum dots (0D) – RMT approach



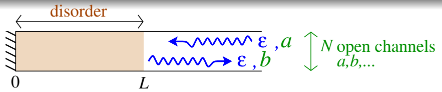
Perfect coupling

- distribution for $N = 1$: Gopar, Mello, Büttiker, PRL, '97
- $\Gamma = \mathcal{Q}^{-1} \in$ Laguerre ensemble, $\mathcal{P}(\Gamma) \propto (\det \Gamma)^{\beta N/2}\, e^{-(\beta/2)\,\mathrm{tr}\{\Gamma\}}$
  Beenakker, Brouwer, Frahm, PRL '97 ; WRM '99
- distribution of $\mathrm{tr}\{\mathcal{Q}\}$ for $N = 2$: Savin, Fyodorov, Sommers, PRE '01
- moments of $\mathrm{tr}\{\mathcal{Q}\} = \sum_a^N \tau_a$: Mezzadri, Simm, '11, '12, CMP '13
- distribution of $\mathrm{tr}\{\mathcal{Q}\}$: CT, Majumdar, PRL '13
- $\langle \mathrm{tr}\{\mathcal{Q}^{n_1}\}\,\mathrm{tr}\{\mathcal{Q}^{n_2}\}\cdots\rangle$: Cunden, Mezzadri, Simm , Vivo ('14, '15, '16)
- distrib. of truncated sum $\sum_{a=1}^{K<N} \tau_a$: Grabsch, Majumdar, CT, JSP '17

Non-ideal couplings

- $\mathrm{Var}(\mathrm{tr}\{\mathcal{Q}\})$: Fyodorov, Sommers, PRL '96; JMP '97
- marginal distribution of e.v. : Sommers, Savin, Sokolov, PRL '01
- distribution of $\Gamma = \mathcal{Q}^{-1}$: $\mathcal{P}(\Gamma)$ is a matrix integral
  Grabsch, Savin, CT, JPA '18
- distribution of $\mathrm{tr}\{\mathcal{Q}\}$: Grabsch, Savin, CT, JPA '18

Wigner-Smith matrix in disordered waveguides



- $N = 1$: Faris, Tsay '94 ; Comtet, CT, JPA '97 ; CT, Comtet, PRL '99 ; Ossipov, Kottos, Geisel, PRB '00 ; ...
- $N > 1$: (RMT) $\Gamma = \mathcal{Q}^{-1} \in$ Laguerre ens. $\mathcal{P}_{L\to\infty}(\Gamma) \propto e^{-\tau_0\,\mathrm{tr}\{\Gamma\}}$
  Beenakker & Brouwer, Physica E '01
- distribution of $\mathrm{tr}\{\mathcal{Q}\}$ for $L \to \infty$:
  (case with $\langle \mathrm{tr}\{\mathcal{Q}\}^n \rangle = \langle \mathrm{tr}\{\Gamma^{-1}\}^n \rangle = \infty$, $\forall\, n > 0$)
  Grabsch & CT, J.Phys.A **49** (2016)
- marginal distribution of e.v. for finite L : Ossipov, PRL (2018)

Q: Statistical properties of the matrix $\mathcal{Q}$ for L finite ?

Main result :

$\mathcal{Q}$ can be represented as an exponential functional of a matrix BM

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Plan

A continuous model

Hamiltonian

H acts on $\psi(x)$, a N-component wave function

$$H\psi(x) = \varepsilon\,\psi(x) \qquad \text{for } H = -\mathbf{1}_N\,\partial_x^2 + \underbrace{V(x)}_{N\times N}$$

Gaussian disorder with :

$$\langle V_{ab}(x)\,V^*_{cd}(x')\rangle = \sigma\,\underbrace{C_{ab,cd}}_{\text{dimensionless}}\,\delta(x-x')$$

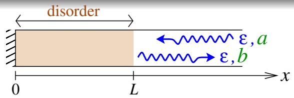
Isotropy assumption

$$C_{ab,cd} = \frac{\beta}{2}\delta_{ac}\delta_{bd} + \left(1-\frac{\beta}{2}\right)\delta_{ad}\delta_{bc} \qquad \begin{cases}\beta = 1 & \text{for } V = V^{\mathrm{T}} \\ \beta = 2 & \text{for } V = V^{\dagger}\end{cases}$$

Elastic m.f.p. ℓ_e & localisation length ξ (for weak disorder)

$$\begin{cases}\ell_e \simeq \frac{2\varepsilon}{\mu\,\sigma} \\ \xi \simeq \frac{8\varepsilon}{\sigma} \sim \beta N \ell_e\end{cases} \qquad \text{with } \mu = 1 + \frac{\beta}{2}(N-1)$$

Matrix wave function



scattering state ↓

$$\psi_b^{(a)}(x) = \frac{1}{\sqrt{2\pi v_\varepsilon}} \left(\delta_{ab}\, e^{-\mathrm{i}k(x-L)} + \mathcal{S}_{ba}(\varepsilon)\, e^{+\mathrm{i}k(x-L)} \right) \quad \text{for } x > L$$

↑ component

with $a,\ b \in \{1, \cdots, N\}$

$$\Psi_\varepsilon(x) = \left(\psi^{(1)}(x) \quad \cdots \quad \psi^{(N)}(x) \right) \text{ is a } N \times N \text{ matrix}$$

obeys $H\Psi_\varepsilon(x) = \varepsilon\, \Psi_\varepsilon(x)$ with $\varepsilon = k^2$ & $v_\varepsilon = 2k$

Matrix-Stochastic Differential Eq. (SDE) for the $\mathcal{S}$-matrix

$$\begin{cases} \sqrt{4\pi k}\, \Psi_\varepsilon(L) = \mathbf{1}_N + \mathcal{S} \\ \sqrt{4\pi k}\, \Psi'_\varepsilon(L) = \mathrm{i}k\left(-\mathbf{1}_N + \mathcal{S}\right) \end{cases}$$

$$\partial_L \mathcal{S} = 2\mathrm{i}k\, \mathcal{S} + \frac{1}{2\mathrm{i}k}\, (\mathbf{1}_N + \mathcal{S})\ V(L)\ (\mathbf{1}_N + \mathcal{S})$$

Matrix-SDE for $\mathcal{S}$ and $\mathcal{Q}$

Study the two matrix-SDE for $\mathcal{S}$ and $\mathcal{Q} = -\mathrm{i}\,\mathcal{S}^\dagger \partial_\varepsilon \mathcal{S}$:

$$\partial_L \mathcal{S} = 2\mathrm{i}k\,\mathcal{S} + \frac{1}{2\mathrm{i}k}\,(\mathbf{1}_N + \mathcal{S})\;V(L)\;(\mathbf{1}_N + \mathcal{S})$$

$$\Downarrow\;\partial/\partial k^2$$

$$\partial_L \mathcal{Q} = \frac{\mathbf{1}_N}{k} + \frac{1}{2\mathrm{i}k}\left[\mathcal{Q}\,V(L)\,(\mathbf{1}_N + \mathcal{S}) - (\mathbf{1}_N + \mathcal{S}^\dagger)\,V(L)\,\mathcal{Q}\right] + \frac{1}{4k^3}(\mathbf{1}_N + \mathcal{S}^\dagger)V(L)(\mathbf{1}_N + \mathcal{S})$$

for $\mathcal{S}(L=0) = \pm\mathbf{1}_N$ and $\mathcal{Q}(L=0) = 0$

Distribution of the matrix $\mathcal{Q}$? Relation to exp. functional of the BM ?

Plan

Start from

$$\int_0^L \mathrm{d}x\, \Psi_\varepsilon^\dagger(x)\Psi_\varepsilon(x) = \frac{1}{2\pi}\left(\mathcal{Q} + \frac{\mathcal{S}-\mathcal{S}^\dagger}{4\mathrm{i}\varepsilon}\right) \simeq \frac{\mathcal{Q}}{2\pi}$$

for metric graphs:CT & Büttiker, PRB**67** (2003)

for the present model: Grabsch & CT, J.Phys.A **53** (2020)

Plausible connection between $\mathcal{Q}$ and exp. functional

Anderson localisation in wave guides

- Spectrum of Lyapunov exponents:

$$\gamma_n = \frac{\sigma}{8k^2}\left(1+\beta\,(n-1)\right) \qquad \text{for } n \in \{1,\ldots,N\}$$

(for weak disorder)

Dorokhov-**M**ello-**P**ereyra-**K**umar theory (1982, 1988)
Beenakker, RMP **69** (1997)

- Spectrum of LE for $\partial_t X(t) = \eta(t)\,X(t)$: $\gamma_n = \frac{\beta}{2}(N-2n+1)$
Newman, CMP **103** (1986)

- "smooth" part of $\Psi_\varepsilon(x)$ obeys

$$\partial_x \widetilde{\Psi}_\varepsilon(x) \simeq \left(\frac{\mu}{\xi} + \frac{1}{\sqrt{\xi}}\,\eta(x)\right)\,\widetilde{\Psi}_\varepsilon(x)$$

with $\frac{1}{\xi} = \frac{\sigma}{8k^2}$ and $\mu = 1 + \frac{\beta}{2}(N-1)$

Plan

Idea of the derivation:

- Identify fast / slow variables : $\mathcal{S} = \overbrace{e^{+2\mathrm{i}kL}}^{\text{rapid}} \underbrace{\widetilde{\mathcal{S}}}_{\text{slow}}$

- The "*square root trick*" : Introduce $\widetilde{\mathcal{S}} = \mathcal{U}_L \mathcal{U}_R$
  symmetrize $\mathcal{Q} = -\mathrm{i}\mathcal{S}^\dagger \partial_\omega \mathcal{S} \Rightarrow \widetilde{\mathcal{Q}} = \mathcal{U}_R\, \mathcal{Q} \mathcal{U}_R^\dagger$

- Decouple $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{Q}}$ (Stratonovich $\to$ Itô & isotropy $\to$ Stratonovich)

Result : a matrix-SDE

$$\frac{\partial}{\partial L}\widetilde{\mathcal{Q}} = \frac{\mathbf{1}_N}{k} - \frac{2\mu}{\xi}\widetilde{\mathcal{Q}} + \frac{1}{\sqrt{\xi}}\left(\widetilde{\mathcal{Q}}\,\eta(L) + \eta(L)\,\widetilde{\mathcal{Q}}\right) \qquad \text{(Stratonovich)}$$

where $\begin{cases} \eta(x) \text{ is a normalized matrix isotropic Gaussian white noise} \\ \mu = 1 + \frac{\beta}{2}(N-1) \end{cases}$

Exponential functional of the matrix BM

matrix-SDE $\longrightarrow$ integral representation :

$$\partial_x \Lambda(x) = [-\mu + \eta(x)]\ \Lambda(x) \qquad \text{(Stratonovich)}$$

$$\boxed{\widetilde{\mathcal{Q}} \overset{\text{(law)}}{=} 2\tau_\xi \int_0^{L/\xi} \mathrm{d}x\, \Lambda(x)^\dagger \Lambda(x)}$$

with $\tau_\xi = \xi/v = \xi/(2k)$

Case $N = 1$

$\Lambda(x) = e^{-\mu\, x + B(x)}$ where $B(x) = \int_0^x \mathrm{d}t\, \eta(t)$ is a BM

$$\widetilde{\mathcal{Q}} \overset{\text{(law)}}{=} 2\tau_\xi \int_0^{L/\xi} \mathrm{d}x\, \Lambda(x)^2 = 2\tau_\xi \int_0^{L/\xi} \mathrm{d}x\, e^{-2\mu\, x + 2B(x)}$$

Distribution of the matrix for $L \to \infty$

The SDE for $\widetilde{\mathcal{Q}}$ is a particular case of a more general mSDE studied in

Grabsch & CT, Europhys. Lett. **116**, 17004 (2016)

$$\widetilde{\mathcal{Q}} \overset{\text{(law)}}{=} 2\tau_\xi \int_0^{L/\xi} \mathrm{d}x\, \Lambda(x)^\dagger \Lambda(x)$$

and $\Gamma = 2\tau_\xi\, \widetilde{\mathcal{Q}}^{-1}$ have a limit law for $L \to \infty$ (if $\mu > \beta(N-1)/2$) :

$$\mathcal{P}_{L\to\infty}(\Gamma) \propto (\det\Gamma)^{\mu-1-\beta(N-1)/2} e^{-\operatorname{tr}\{\Gamma/2\}} \;\to\; \mathcal{P}_\infty(\Gamma) \propto (\det\Gamma)^{0} e^{-\operatorname{tr}\{\Gamma/2\}}$$

Wishart distribution

Wigner-Smith time-delay matrix : $\mu = 1 + \beta(N-1)/2$

Comparison with known result

Quantum scattering + RMT $\Rightarrow$ distribution of $\Gamma = \widetilde{\mathcal{Q}}^{-1}$ is

$$\mathcal{P}^{(\mathrm{BB})}(\Gamma) \propto e^{-\tau_0 \operatorname{tr}\{\Gamma\}}$$

Beenakker & Brouwer, Physica E **9** (2001)

Eigenvalues of $\mathcal{Q}$ for L finite

$$\widetilde{\mathcal{Q}} \overset{\text{(law)}}{=} 2\tau_\xi \int_0^{L/\xi} \mathrm{d}x\, \Lambda(x)^\dagger \Lambda(x) \qquad \text{\& matrix SDE for } \widetilde{\mathcal{Q}}$$

Deduce moments

$$\left\langle \operatorname{tr}\{\mathcal{Q}\}^2 \right\rangle = \frac{N\tau_\xi^2}{2} \left\{ \frac{1+\frac{\beta N}{2}}{1+\frac{\beta}{2}} \left[e^{4L/\xi} - 1 - \frac{4L}{\xi} \right] + \frac{\left(\frac{2}{\beta}\right)^2 (N-1)}{1+\frac{\beta}{2}} \left[e^{-2\beta L/\xi} - 1 + \frac{2\beta L}{\xi} \right] \right\}$$

$$\left\langle \operatorname{tr}\left\{\mathcal{Q}^2\right\} \right\rangle = \frac{N\tau_\xi^2}{2} \left\{ \frac{1+\frac{\beta N}{2}}{1+\frac{\beta}{2}} \left[e^{4L/\xi} - 1 - \frac{4L}{\xi} \right] - \frac{\frac{2}{\beta} (N-1)}{1+\frac{\beta}{2}} \left[e^{-2\beta L/\xi} - 1 + \frac{2\beta L}{\xi} \right] \right\}$$

Eigenvalues (proper time delays) :

$$\langle \tau_a \rangle = \frac{L}{k} = 2\tau_\xi \frac{L}{\xi} \qquad \left(\leftrightarrow \text{DoS } \langle \rho_L(\varepsilon) \rangle = \frac{1}{2\pi} \sum_a \langle \tau_a \rangle = \frac{NL}{2\pi\sqrt{\varepsilon}}\right)$$

$$\langle \tau_a^2 \rangle \simeq \tau_\xi^2 \frac{N\beta}{2(\beta+2)} e^{4L/\xi}, \qquad \langle \tau_a \tau_b \rangle = \tau_\xi^2 \frac{4}{\beta} \left[\frac{L}{\xi} + \frac{e^{-2\beta L/\xi} - 1}{2\beta} \right] \simeq \tau_\xi^2 \frac{4L}{\beta\xi} \quad \text{for } a \neq b$$

(eigenvalues are *anti*-correlated)

Plan

Scattering pb in multichannel wires : $\mathcal{S}(\omega)$ and $\mathcal{Q}(\omega) = -\mathrm{i}\,\mathcal{S}^\dagger\,\partial_\omega\mathcal{S}$

- Square root trick and symmetrisation

$$\Rightarrow \widetilde{\mathcal{S}} = \mathcal{U}_L\mathcal{U}_R \text{ and } \widetilde{\mathcal{Q}} = \mathcal{U}_R\,\mathcal{Q}\,\mathcal{U}_R^\dagger = \mathbf{1}_N\,L/k - \mathrm{i}\,\mathcal{U}_L^\dagger\partial_\omega\big(\mathcal{U}_L\mathcal{U}_R\big)\,\mathcal{U}_R^\dagger$$

Remark : this is a different symmetrisation than the one used for QD

$$\mathcal{Q}_s = -\mathrm{i}\,\mathcal{S}^{-1/2}\,\partial_\omega\mathcal{S}\,\mathcal{S}^{-1/2}$$

(Brouwer, Frahm, Beenakker '97)

- Isotropy allows to decouple $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{Q}}$

  get the matrix-SDE

$$\partial_L\widetilde{\mathcal{Q}} = \frac{\mathbf{1}_N}{k} - \frac{2\mu}{\xi}\widetilde{\mathcal{Q}} + \frac{1}{\sqrt{\xi}}\left(\widetilde{\mathcal{Q}}\,\eta(L) + \eta(L)\,\widetilde{\mathcal{Q}}\right)$$

for $\mu = 1 + \beta(N-1)/2$

- Functional exponential for matrix BM

$$\widetilde{\mathcal{Q}} \overset{\text{(law)}}{=} 2\tau_\xi \int_0^{L/\xi} \mathrm{d}x\,\Lambda(x)^\dagger\Lambda(x) = 2\tau_\xi\, Z_{L/\xi}^{(\mu)}$$

Exponential functionals of the BM & Kesten variables

$$Z_L^{(\mu)} = \int_0^L \mathrm{d}x\, e^{-2\mu\, x + 2\, B(x)} \qquad \Rightarrow \text{ much studied in Mathematics}$$

M. Yor, *Exponential Functionals of BM and Related Processes*, Springer, 2000, ...

- Discrete version : Kesten variable Kesten, Acta. Math. **131** (1973)

  ξ_i : i.i.d. random variables (> 0) with distribution $w(\xi)$

$$Z_\infty = \xi_1 + \xi_1\xi_2 + \xi_1\xi_2\xi_3 + \cdots = \sum_{n=1}^{\infty}\prod_{i=1}^{n}\xi_i$$

  Tail with "disorder" dependent exponent

$$\mathscr{P}(Z) \sim Z^{-1-\mu} \text{ for } \langle \xi^\mu \rangle = 1$$

  de Calan, Luck, Nieuwenhuizen & Petritis, J.Phys.A **18** (1985)

- Application for risk theory

Scand. Actuarial J. 1990: 39–79

The Distribution of a Perpetuity, with Applications to Risk Theory and Pension Funding

By Daniel Dufresne, Université de Montréal

Matrix generalizations :

- continuous:

  Rider & Valko, Int. Math. Res. Not. (2016)

  A. Grabsch & C. Texier, J. Phys. A (2020)

- discrete:

  Gautié, Bouchaud & Le Doussal, J.Phys.A (2021)

  Arista, Bisi & O'Connell, arXiv:2112.12558 (2022)

Wigner-Smith time delay matrix for finite L

Open questions :

- Correlation functions: we conjecture

  $$\langle \mathrm{tr}\{\mathcal{Q}^{n_1}\}\,\mathrm{tr}\{\mathcal{Q}^{n_2}\}\cdots\mathrm{tr}\{\mathcal{Q}^{n_k}\}\rangle \sim e^{2n(n-1)L/\xi} \qquad \text{where } n=\sum_{i=1}^{k} n_i$$

- $\mathcal{P}_L(\widetilde{\mathcal{Q}}) = ?$
  [known for $N=1$, cf. Comtet & CT, J.Phys.A **30** (1997)]

- Beyond isotropy assumption ?
  Channels with different velocities ?

Thank you!

A. Grabsch & C. Texier, J. Phys. A: Math. Theor. **53**, 425003 (2020)