Multifractal Properties Of The CFS Peak

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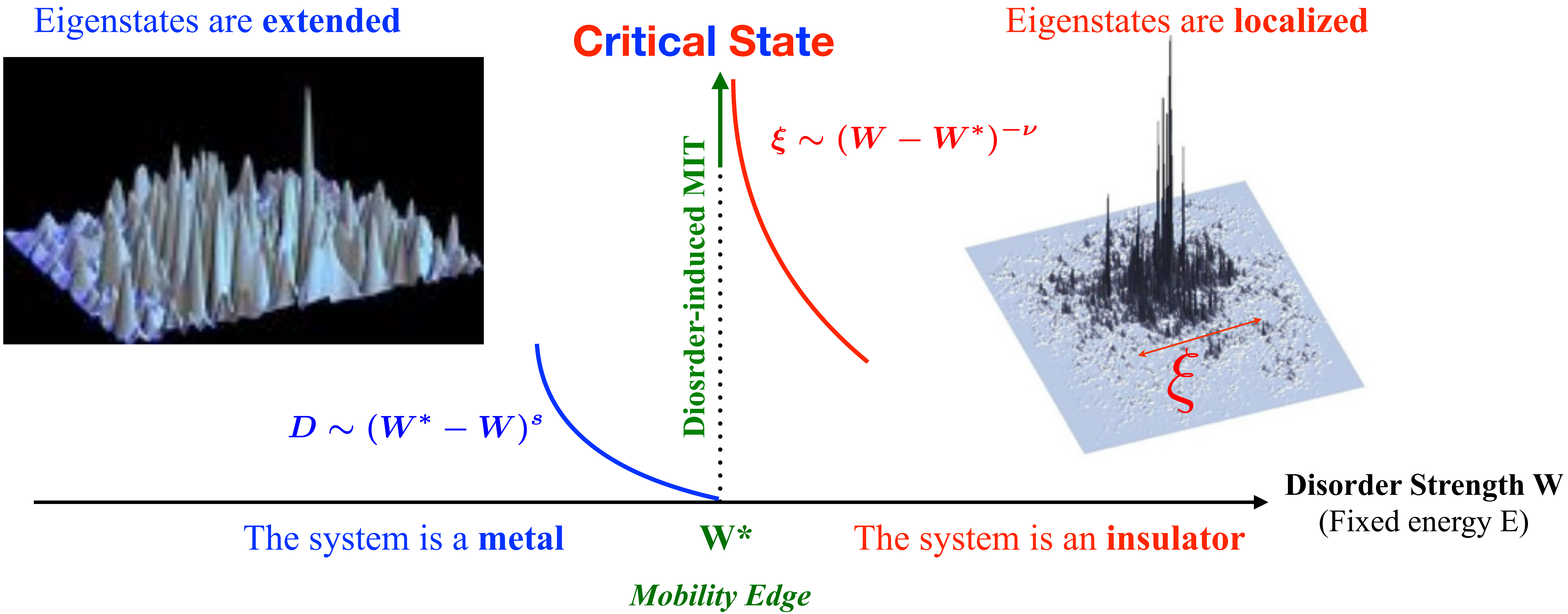
3D Anderson Metal-Insulator Transition (infinite-size disordered systems)

Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + \boxed{V(\mathbf{r})}$ Random potential

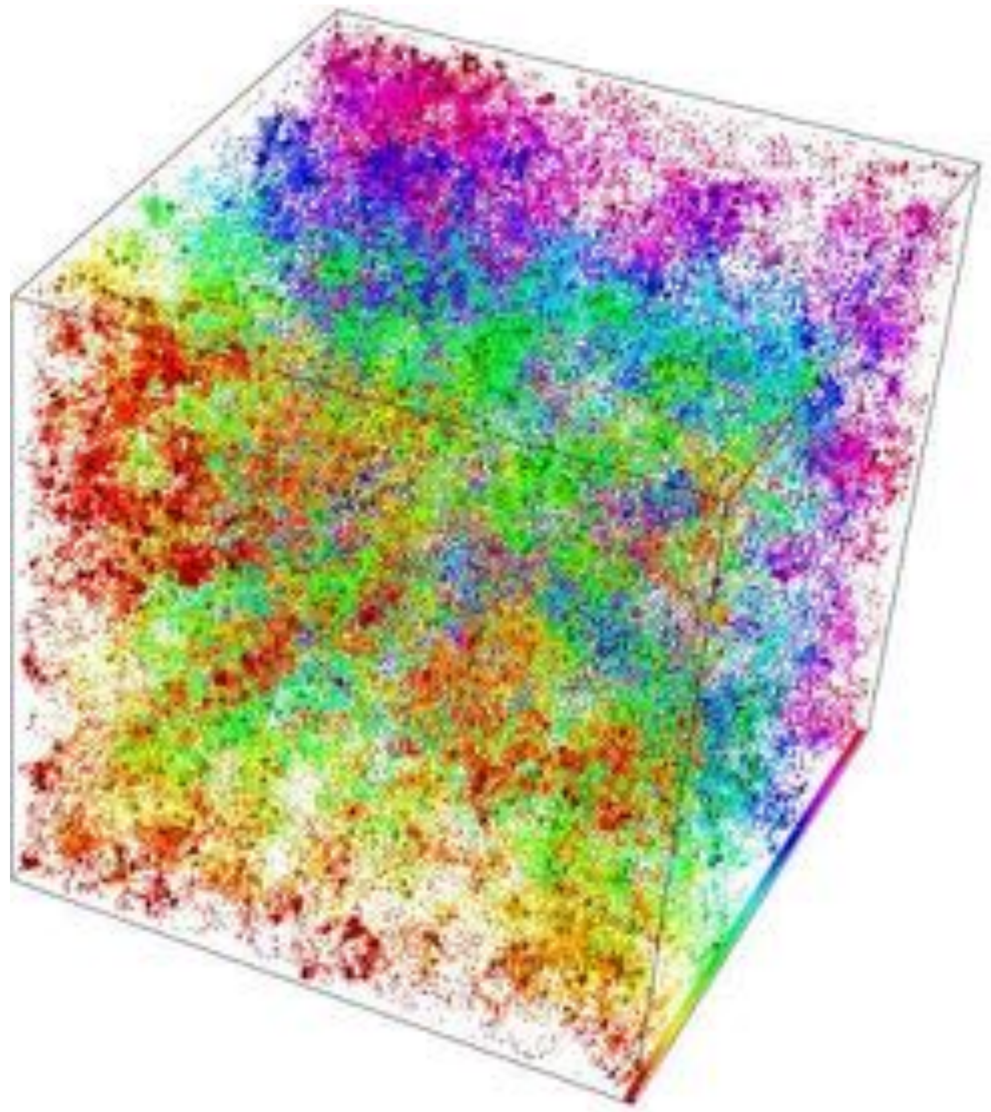
Optical laser Speckle, Uncorrelated box distribution, Continuous or lattice model, etc.



Eigenfunction Fluctuations and how they fill real space

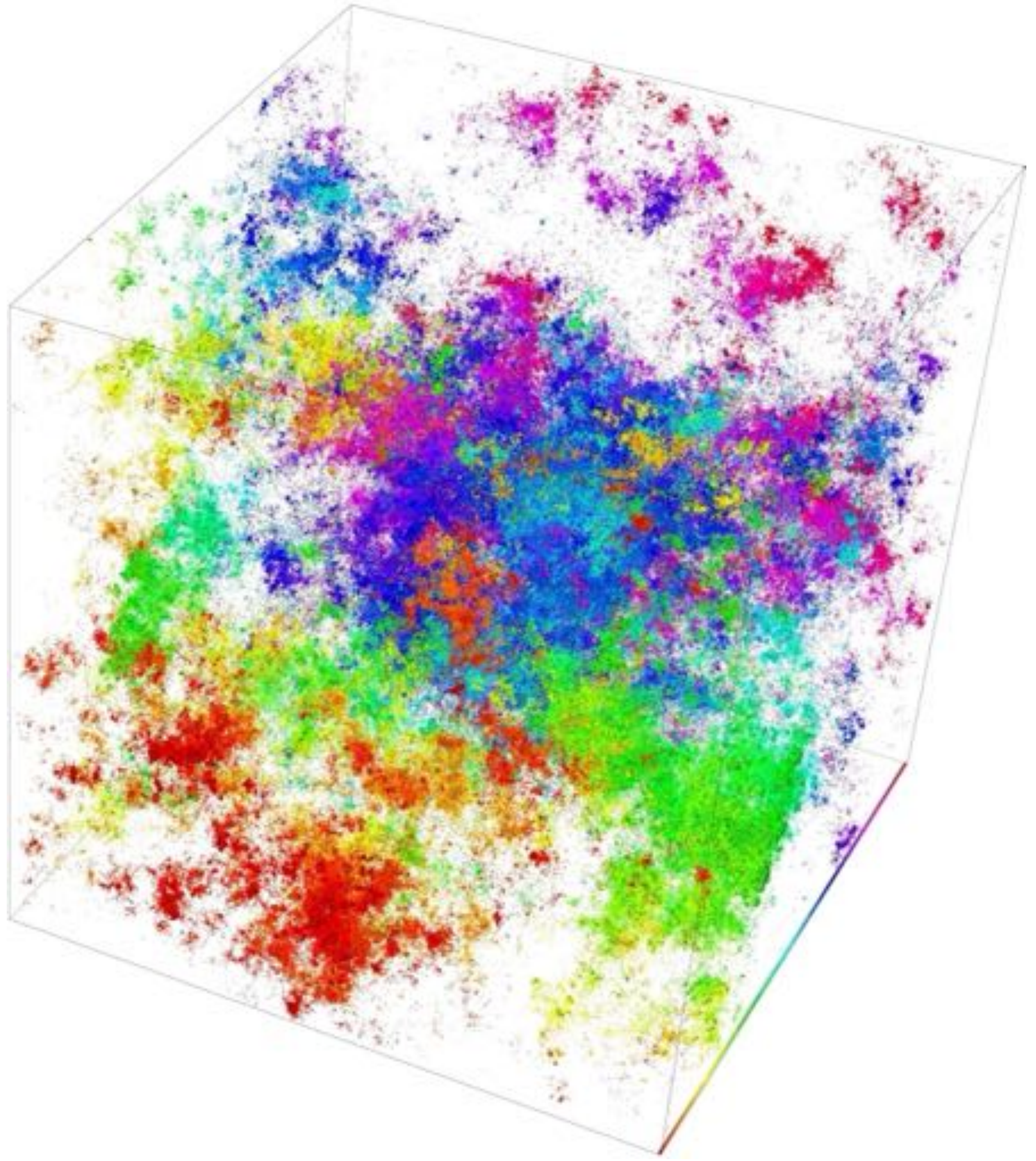
• Metallic Phase:
Extended eigenfunctions

$$|\psi_n|^2 \sim L^{-d}$$



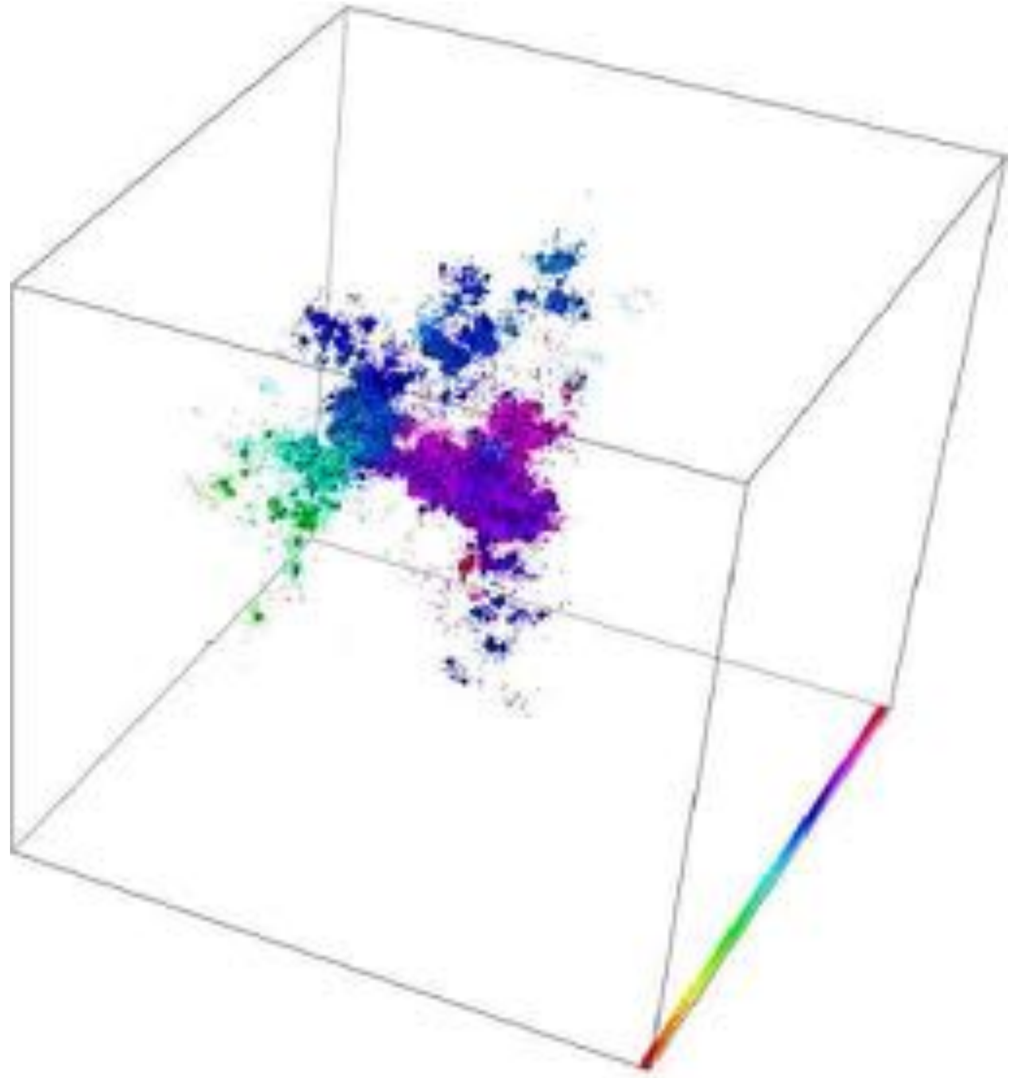
• Mobility Edge:
Fractal eigenfunctions

Strong fluctuations:
Regions where the eigenfunction is exceptionally large, regions where it is exceptionally small



• Insulator:
Localized eigenfunctions

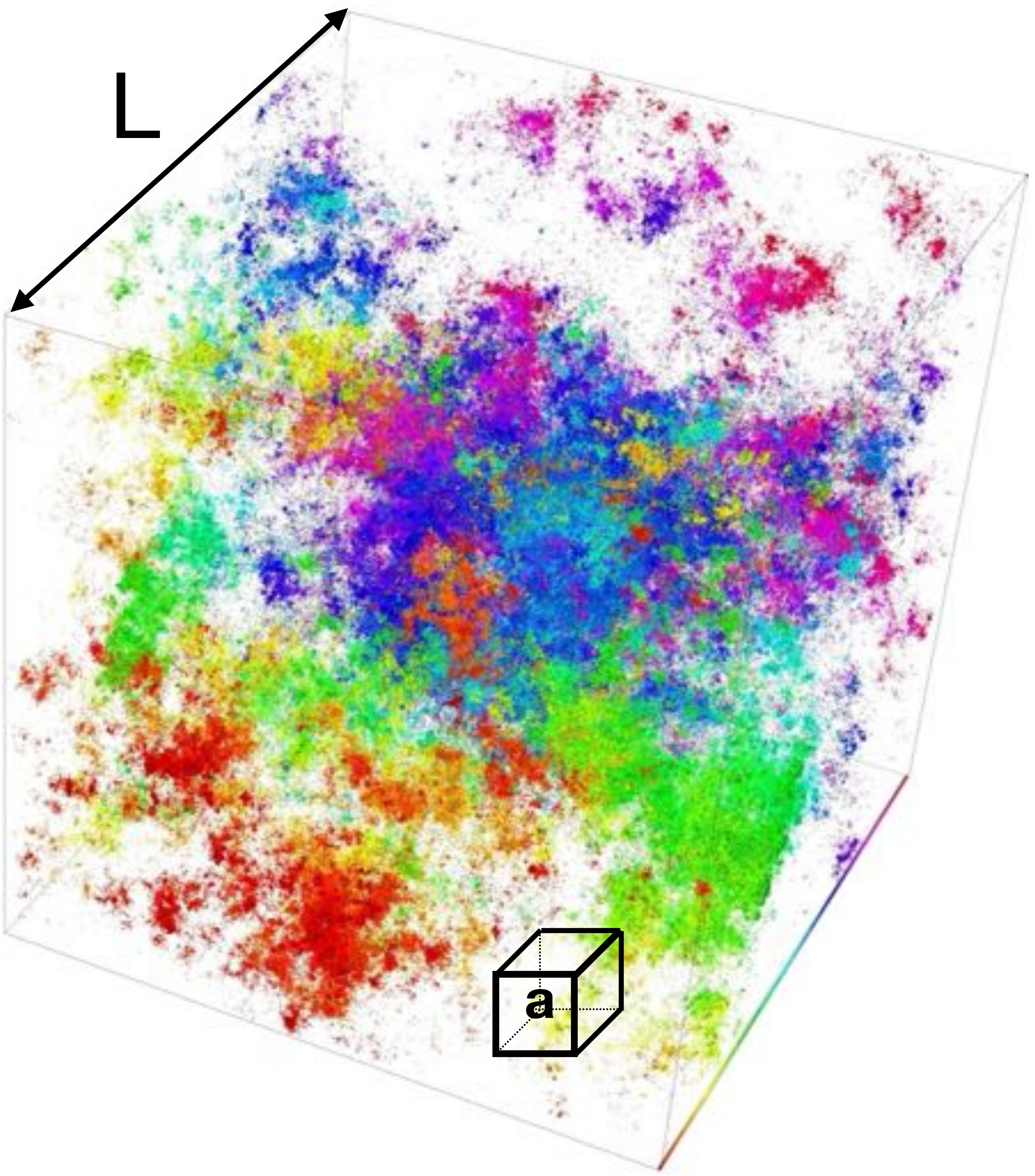
$$|\psi_n|^2 \sim \xi^{-d} e^{-|\vec{r}-\vec{r}_0|/\xi}$$



Rodriguez et. al. PRB 84, 134209 (2011)

How to quantify the intensity fluctuations?

Multifractal Dimensions



Box Counting Method borrowed from Fractal Analysis

$$N = (L/a)^d \text{ boxes } B_j$$ Partition the system in boxes

$$\mu_j = \sum_{i \in B_j} |\psi_i|^2$$

Coarse-grain the wave function intensity on a scale a < L.
Bin intensities in each box.
Defines a probability measure.

Av-GIPR $R_q = \sum_j \overline{\mu_j^q} \sim N^{-(q-1)D_q}$

Multifractal dimension (↑ D_q)

Metal: $D_q = d$

Insulator: $D_q = 0$

Critical state: Non trivial $0 \leq D_q \leq d$

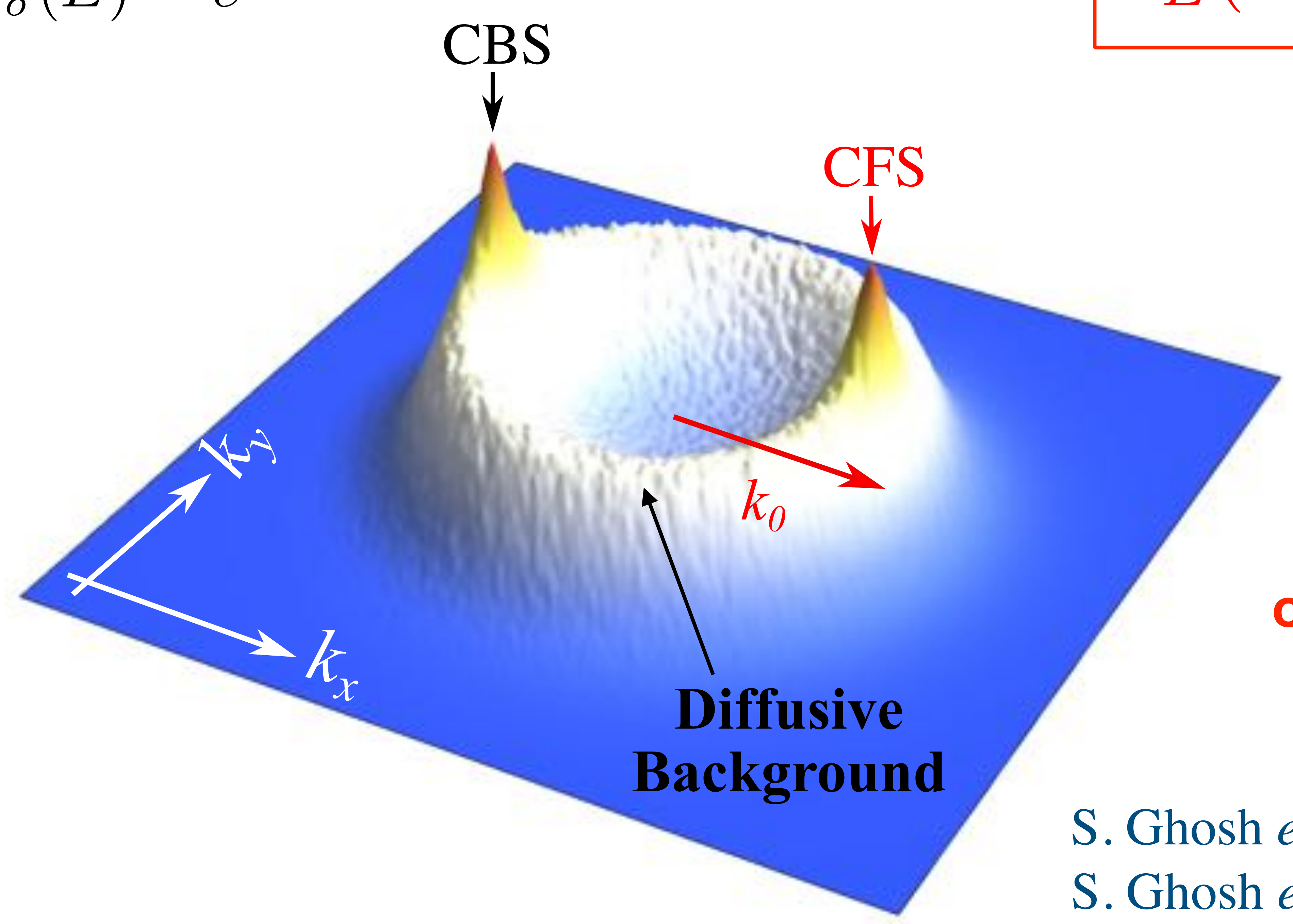
Multifractal wave functions

Rodriguez et. al. PRB ***84****, 134209 (2011)*

CFS peak in momentum space reveals signatures of multifractality in space

Initial atomic cloud with a kick	Time evolution in random medium	Experimentally feasible	"Time-of-flight" measurement	**Disorder-averaged momentum distribution at fixed energy**
(1) Start with an initial plane wave	(2) Propagate with the Hamiltonian ***H***	(3) Filter around energy ***E***	(4) Compute the momentum density	(5) Repeat over many disorder configurations
$\|\psi(t=0)\rangle = \|\boldsymbol{k}_0\rangle$	$\|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}\|\mathbf{k}_0\rangle$	$\|\psi_E(t)\rangle = \hat{F}_\sigma(E)\|\psi(t)\rangle$ $\hat{F}_\sigma(E) = e^{-\frac{(E-\hat{H})^2}{2\sigma^2}}$	$\|\langle\mathbf{k}\|\psi_E(t)\rangle\|^2$	$n_E(\mathbf{k},t) = \overline{\|\langle\mathbf{k}\|\psi_E(t)\rangle\|^2}$

In presence of V(***r***), initial wave spreads over a broad range of energies E (cf ***spectral function***). As a result, properties depending sharply on E are blurred by the energy spread. Need for *energy-filtering*.

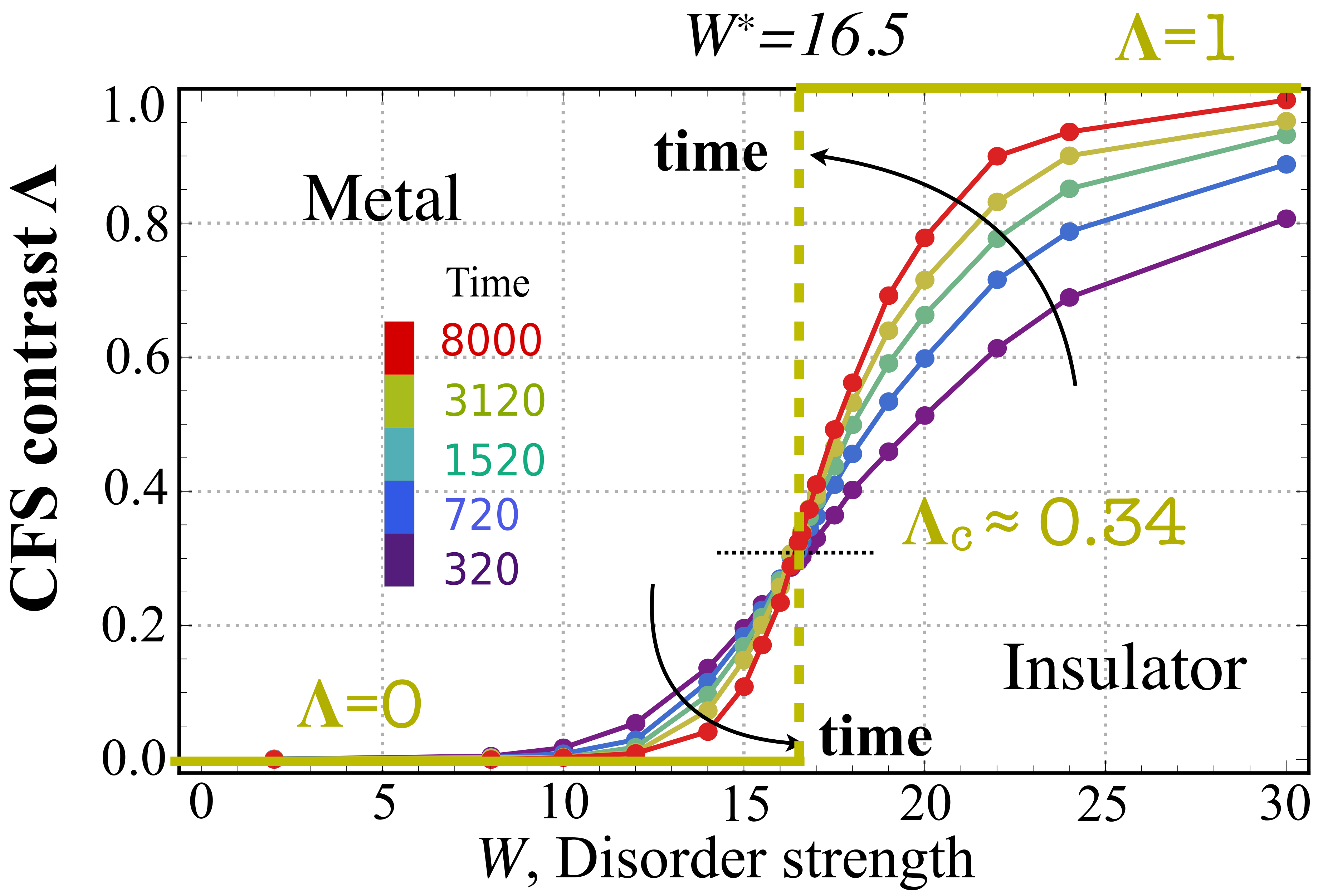


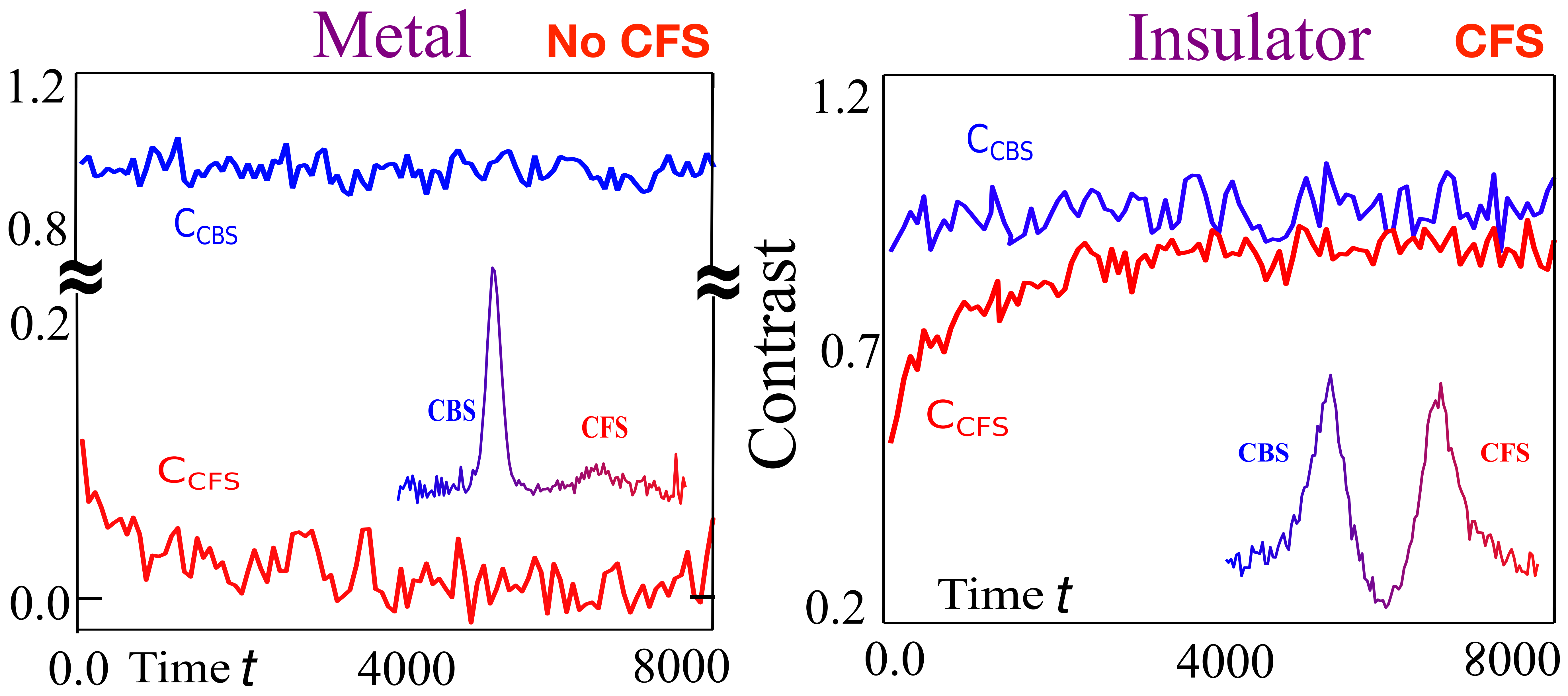
CBS and CFS peaks embodies signatures of localisation transition

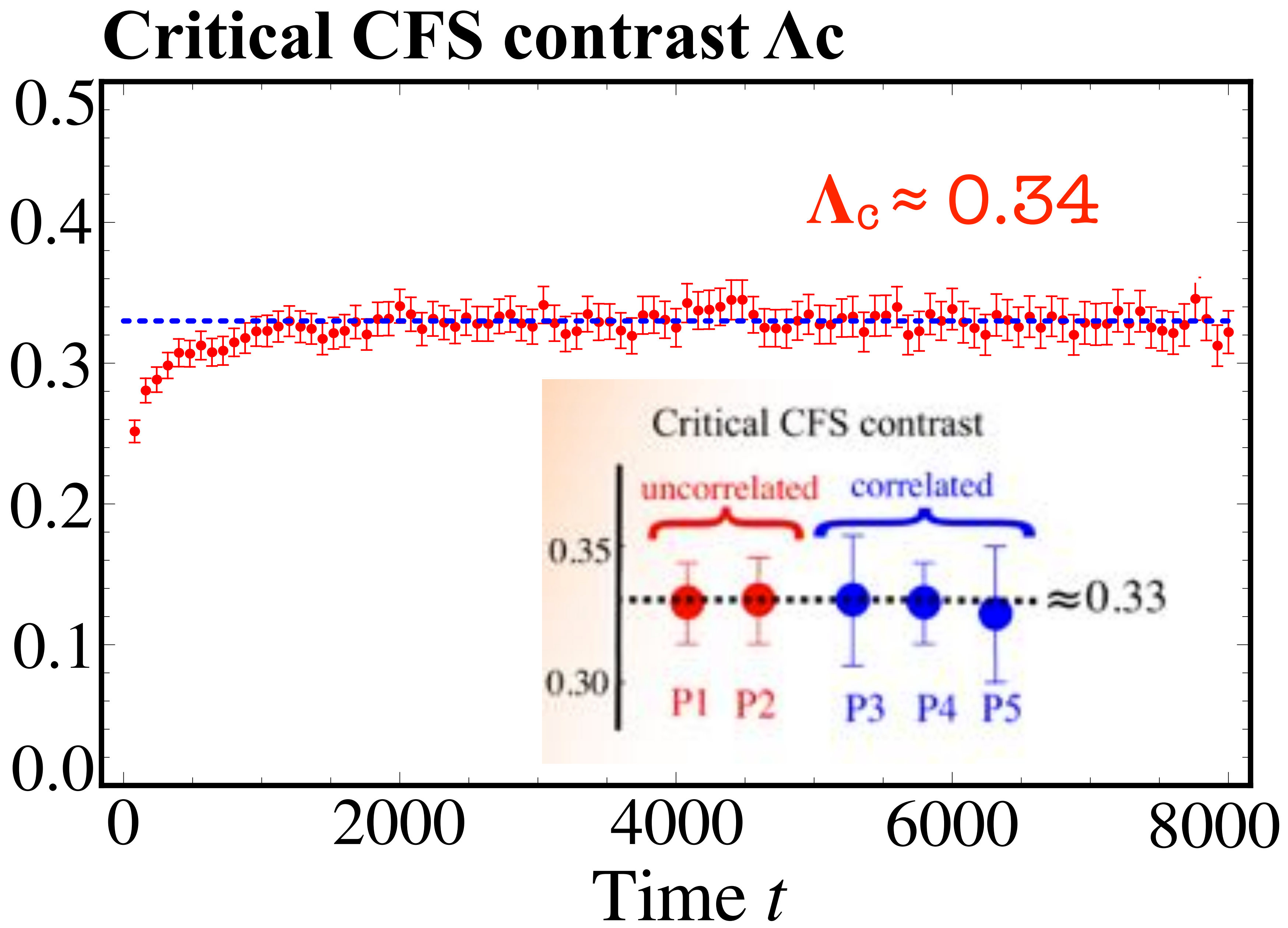
$$\text{CFS Contrast } \Lambda = \frac{\text{Peak Height - Background}}{\text{Background}}$$

S. Ghosh *et al*., PRL **115**, 200602 (2015)
S. Ghosh *et al*., PRA **95**, 041602 (2017)

CFS is a critical quantity!







CFS Contrast CV to a step-function as t increases:
It is a Smoking Gun of 3D AT and a critical quantity!

- Jumps from 0 to 1 across the transition.
- Crossing locates ME.
- CFS takes on an intermediate value at critical point!

S. Ghosh et al., Phys. Rev. A **95**, 041602(R) (2017).

Bogomolny-Giraud Conjecture (2010) & CFS Critical Contrast

$$\overline{\rho}(E) = \frac{1}{N^d}\overline{\sum_n \delta(E - E_n)}$$

Disorder-averaged DoS per unit volume

$$K_E(t) = \frac{1}{N^d}\overline{\sum_{n,m} e^{-i(E_n - E_m)t}}\Big|_E$$

Spectral Form Factor

$$E = \frac{E_n + E_m}{2}$$

$$\boxed{\tau_H = 2\pi\, N^d\, \overline{\rho}(E)}$$

Heisenberg Time (time scale associated to the mean-level spacing)

Spectral Rigidity $\Sigma_2 = \overline{N^2} - \overline{N}^2$ **Level number fluctuations in an energy interval**

Spectral Compressibility $\Sigma_2 \sim \kappa\, \overline{N}$ for $\overline{N} \gg 1$

$$\boxed{\kappa = 2\pi\hbar\, N^d\, \overline{\rho}(E)\, K_E(t \to 0^+)}$$

$$K_E(t \to 0^+) = \lim_{\substack{(t,N)\to\infty \\ t/\tau_H \to 0}} K_E(t)$$

Bogomolny-Giraud

$$\boxed{\kappa = 1 - D_1/d}$$

$\lim_{q\to 1} D_q = D_1$ **Information dimension**

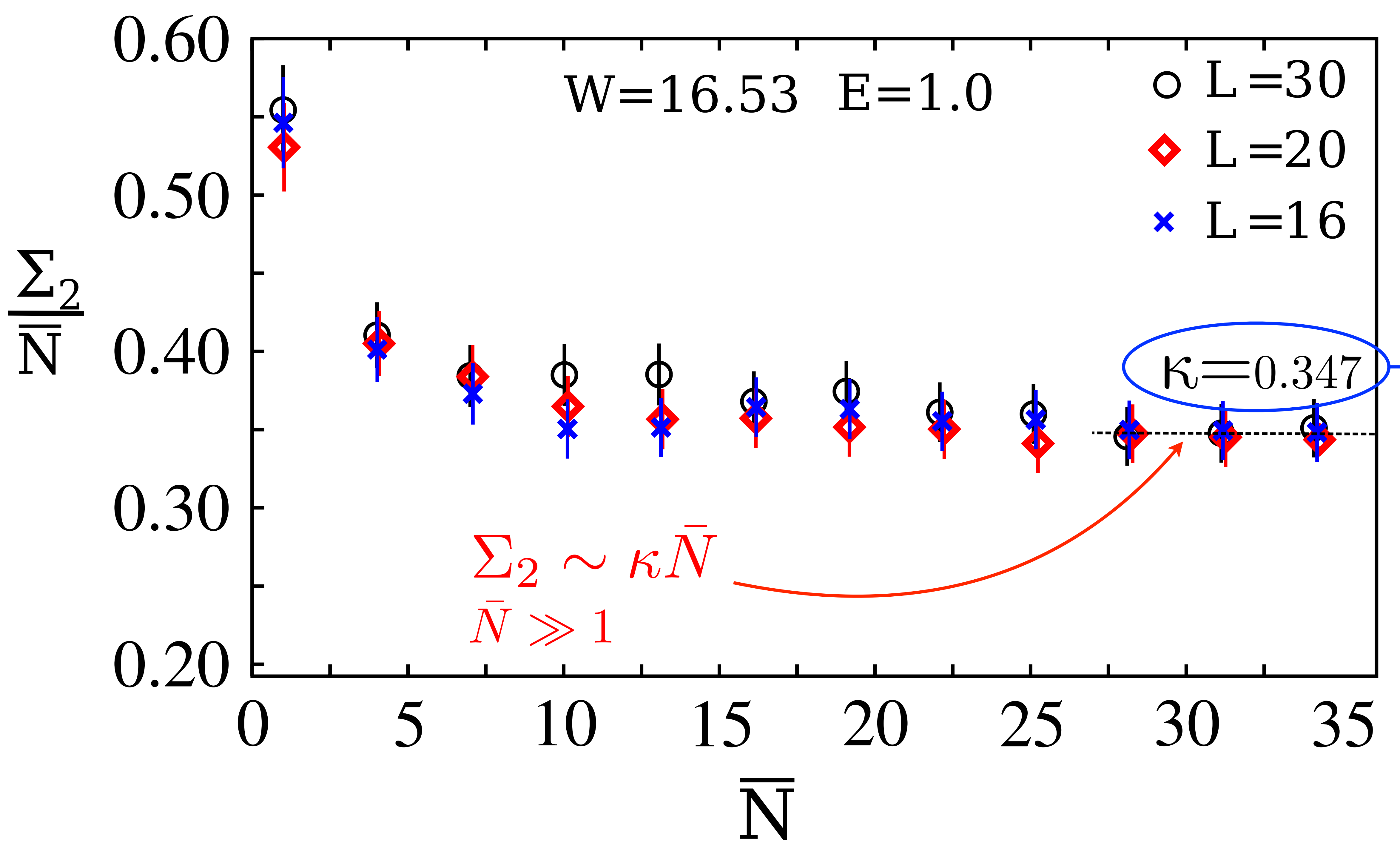
CFS contrast

$$\Lambda(t) = 2\pi\hbar\, N^d\, \overline{\rho}(E)\, K_E(t)$$

At critical point: $\Lambda(t) = \Lambda_c = \text{Cte}$

We infer

$$\Lambda_c = \kappa$$



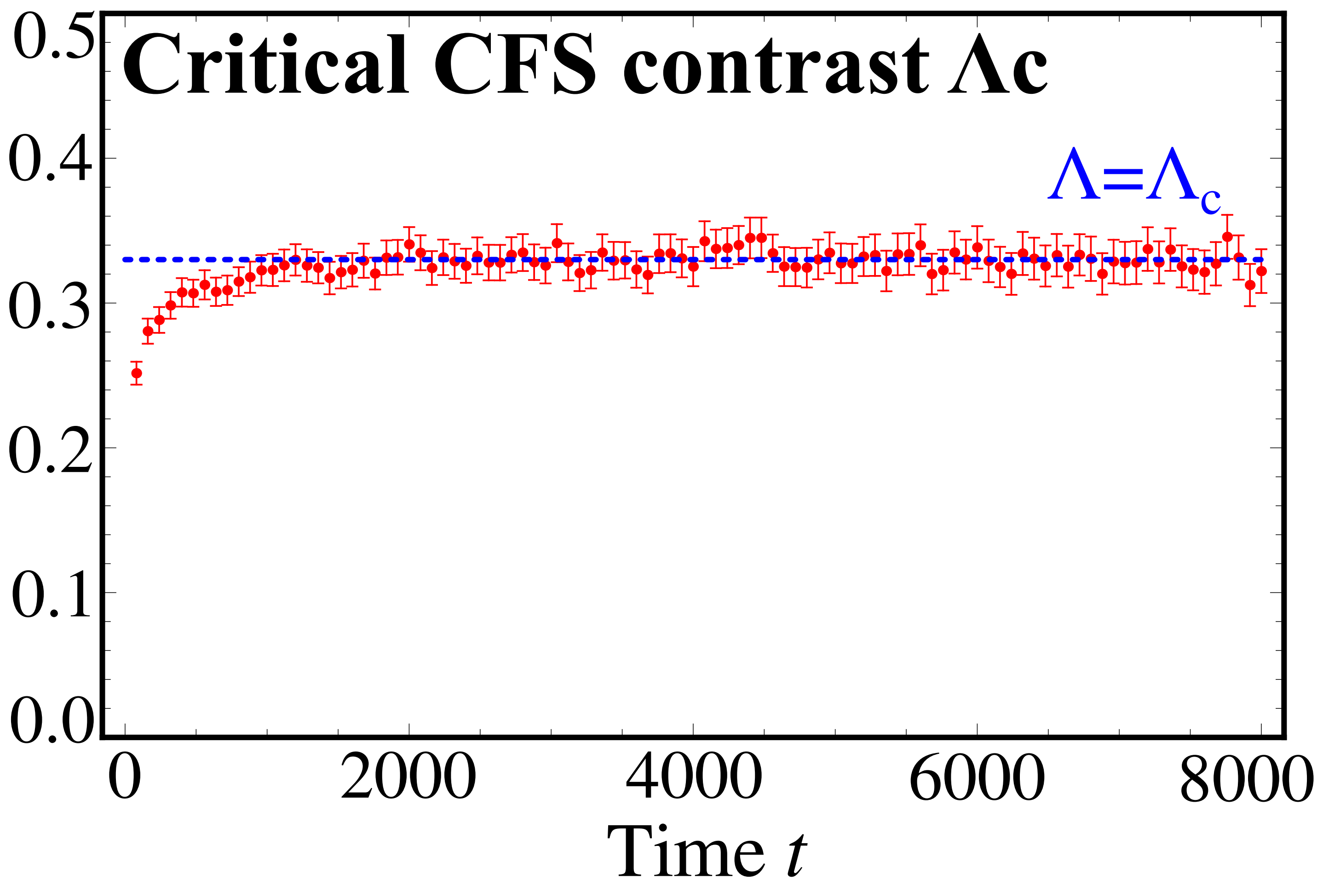
To be compared to the numerically computed critical CFS contrast $\Lambda_c \approx 0.342 \pm 0.01$

Numerically $\Lambda_c \approx 0.342 \pm 0.01 \Rightarrow$ $\boxed{D_1 \approx 1.974 \pm 0.03}$ **To be compared to** $D_1 = 1.958$

S. Ghosh et al., PRA **95**, 041602 (2017)

Rodriguez *et al.*, PRB **84**, 134209 (2001)

But what is the interplay between time and system size?



$$\Lambda(t) = \Lambda_c = \text{Cte}$$

$$\boxed{\Lambda(t) = 2\pi\hbar\, N^d\, \overline{\rho}(E)\, K_E(t)} \qquad K_E(t \to 0^+)$$

Independent of time at critical point (after a while) … *but what time?*

KEY POINT: $\lim_{t\to\infty} \lim_{N\to\infty} (\cdots) \neq \lim_{N\to\infty} \lim_{t\to\infty} (\cdots)$ **The two limits DO NOT commute!**

At the critical point, no spatial scale $\Longrightarrow$ **The only relevant time scale is the one associated to the system size N. It is the Heisenberg time τ_H**

There are thus two regimes to understand:

$t \ll \tau_H$ — **Infinite size limit**

$t \gg \tau_H$ — **Infinite time limit**

Critical models with multifractal eigenstates used here:

- Kicked Systems ⟶ Floquet Quantum Maps: $U = e^{-i\varphi_{\mathbf{p}}}\, e^{-iV(\mathbf{r})}$ Uniformly distributed over 2π

3D Random Kicked Rotor (3DKR) $V(\mathbf{r}) = K\, v(x)\, v(y)\, v(z)$ $\quad v(x) = \frac{\sqrt{2}}{4}(2\cos x + \sin 2x)$

Exhibits an Anderson transition at K = 1.58

Flat DoS

$$D_1 = 1.912 \qquad D_2 = 1.165$$

Ruijsenaars-Schneider (RS) $V(\mathbf{r}) = a\, x \qquad x \in [-\pi, \pi]$ $\quad D_q(a)$

Multifractal eigenstates, flat DoS

D_q indepepent on quasi-energy

These parameters control multifractality

- Time-independent Hamiltonian: $G(\mu, \sigma)$ Gaussian distribution of mean μ and dispersion σ

Power-law Random Banded Matrices

Diagonal entries are i.i.d. with distribution G(0, 1)

Real and imaginary parts of the off-diagonal entries are i.i.d. with G(0, σ_{mn})

$$\sigma_{mn}^{-2} = 1 + \frac{\sin^2(\pi|n-m|/N)}{(b\pi/N)^2}$$

Multifractal eigenstates

D_q and DoS depend on E

$$D_q(E, b)$$

Momentum Distribution - Technicalities

$$n(\mathbf{k},t) = \frac{1}{N^d}\overline{|\langle \mathbf{k}|U(t)|\mathbf{k}_0\rangle|^2} \qquad U(t) = e^{-iHt} \text{ or a quantum map}$$

- **Eigenfunction/Eigenenergy decomposition:** $H|\phi_n\rangle = E_n|\phi_n\rangle$

(or Floquet eigenstates and quasi-energies)

$$U|\phi_n\rangle = e^{iE_n}|\phi_n\rangle$$

$$n(\mathbf{k},t) = \frac{1}{N^d}\overline{\sum_{n,m} e^{-i(E_n-E_m)t}\,\phi_n(\mathbf{k})\phi_n^*(\mathbf{k}_0)\phi_m(\mathbf{k}_0)\phi_m^*(\mathbf{k})}$$

- **Momentum Distribution at fixed energy E:**

$$n(\mathbf{k},t) = \int n(\mathbf{k},t;E)\,dE$$

$$n(\mathbf{k},t;E) = \frac{1}{N^d}\overline{\sum_{n,m} e^{-i(E_n-E_m)t}\,\phi_n(\mathbf{k})\phi_n^*(\mathbf{k}_0)\phi_m(\mathbf{k}_0)\phi_m^*(\mathbf{k})\,\delta(E-\frac{E_n+E_m}{2})}$$

Interference-Free Classical (Diffusive) Background and Relation to Spectral Function

$$A(\mathbf{k}, E) = \frac{1}{N^d} \overline{\sum_n \delta(E - E_n)\, |\phi_n(\mathbf{k})|^2}$$

Spectral Function

Sum Rules:

$$\frac{1}{N^d} \sum_{\mathbf{k}} A(\mathbf{k}, E) = \rho(E) \qquad \int A(\mathbf{k}, E)\, dE = 1$$

Average momentum distribution at E $\equiv \dfrac{A(\mathbf{k}, E)}{\rho(E)}$

$A(\mathbf{k}, E) \equiv$ Probability density to have energy E at momentum **k**

$$n_D(\mathbf{k}) = \int \frac{A(\mathbf{k}, E)}{\rho(E)} A(\mathbf{k}_0, E)\, dE$$

$$n_D(\mathbf{k}; E) = \frac{A(\mathbf{k}, E)}{\rho(E)} A(\mathbf{k}_0, E)$$

Exact result (Ergodicity)

Absence of correlations between norm and phase of the eigenstates **in direct space**

$$\Longrightarrow \quad A(\mathbf{k}, E) \approx \rho(E) \quad \Longrightarrow \quad n_D(\mathbf{k}; E) \approx \rho(E)$$

Numerically-approved: Works well for the models explored here!

CFS Contrast in the Long-Time Limit at Finite Size

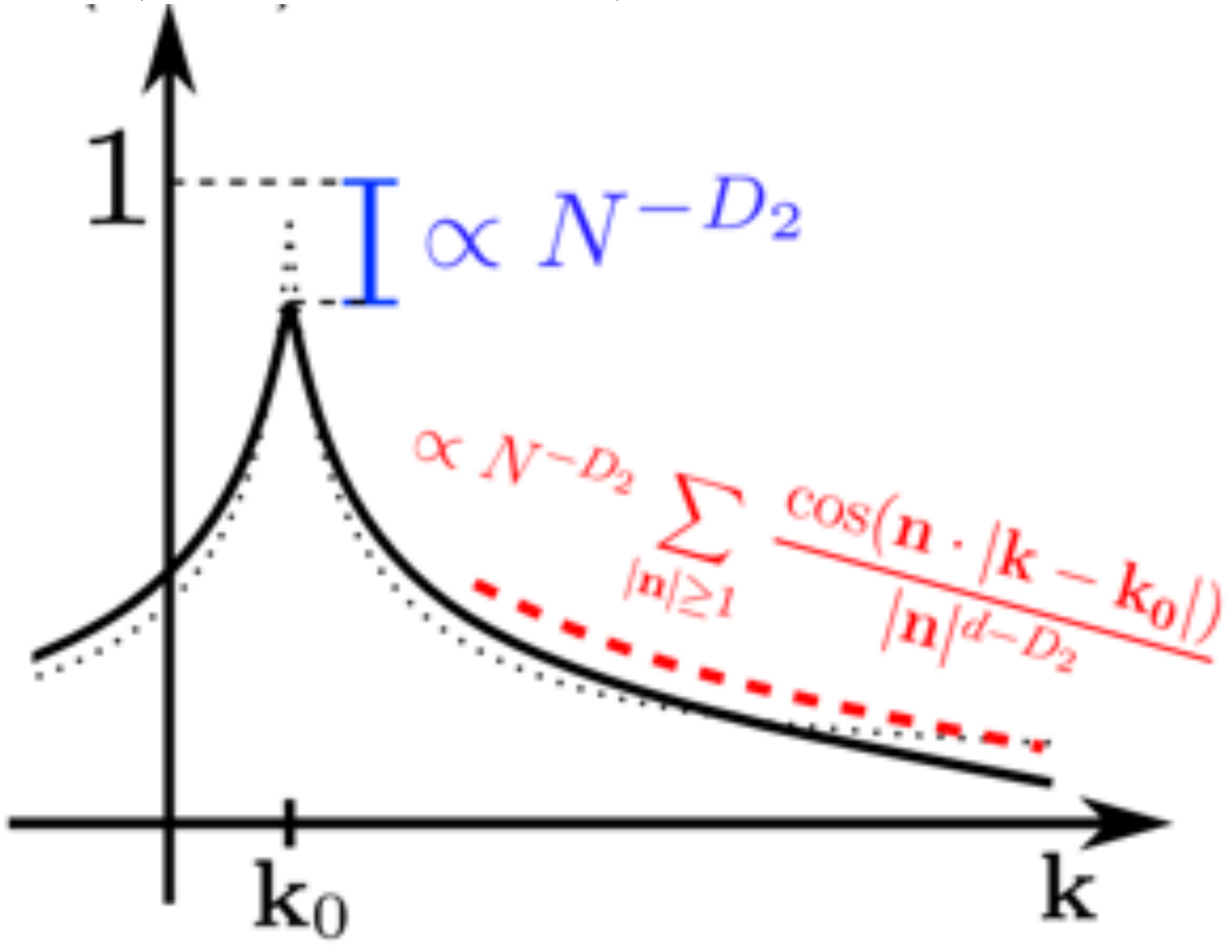
$t \gg \tau_H$

$$\Lambda_N(\mathbf{k},t;E) = \frac{n(\mathbf{k},t;E) - n_D(\mathbf{k};E)}{n_D(\mathbf{k};E)} = \frac{n(\mathbf{k},t;E)}{\rho(E)} - 1$$

$$n(\mathbf{k},t;E) = \frac{1}{N^d}\overline{\sum_n |\phi_n(\mathbf{k})|^2|\phi_n(\mathbf{k}_0)|^2\,\delta(E-E_n)} + \overline{\sum_{n\neq m} e^{-i(E_n-E_m)t}\,\phi_n(\mathbf{k})\phi_n^*(\mathbf{k}_0)\phi_m(\mathbf{k}_0)\phi_m^*(\mathbf{k})\delta(E-\frac{E_n+E_m}{2})}$$

$t \to \infty$

$\Lambda_N(\mathbf{k}, t=\infty; E)$



$$\Lambda_N(\mathbf{k}_0, t=\infty; E) = \frac{1}{N^d\rho(E)}\overline{\sum_n |\phi_n(\mathbf{k}_0)|^4\,\delta(E-E_n)} - 1$$

Fourier Transform
$\phi_n(\mathbf{k}) \to \phi_n(\mathbf{r})$

$$\Lambda_N(\mathbf{k}_0, t=\infty; E) = 1 - \frac{\overline{\sum_\mathbf{r} |\phi_n(\mathbf{r})|^4\,\delta(E-E_n)}}{\overline{\sum_n \delta(E-E_n)}}$$

IPR $\sim N^{-D_2}$

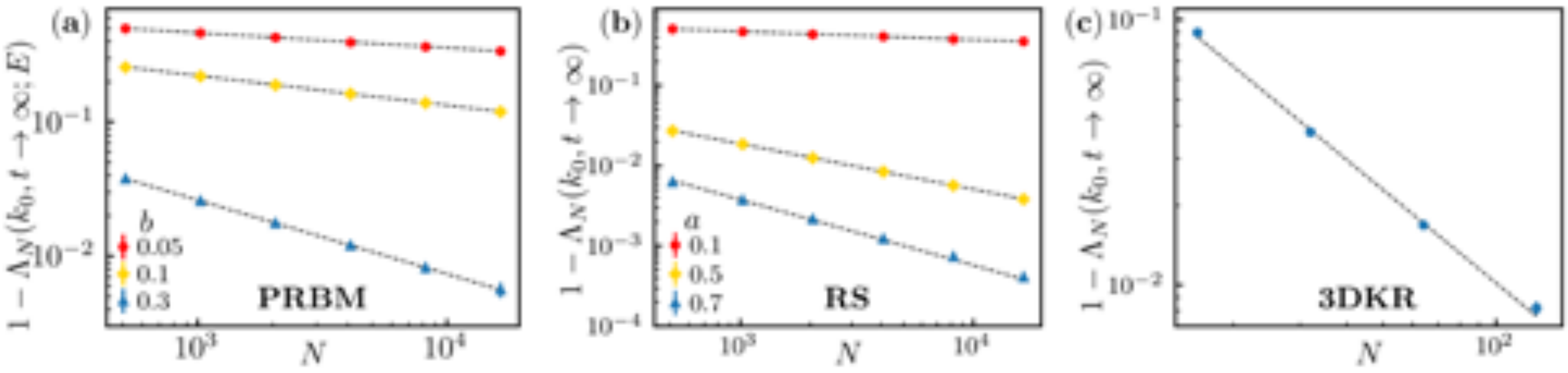


Figure 3. CFS contrast peak in the long-time limit ($t \gg \tau_H$) and its scaling (56) with system size N. (a) PRBM model with different b and at $E = 0$. (b) RS model averaged over E with different a. (c) 3DKR model with $K = 1.58$. Symbols are numerical data for different system sizes. Dashed black lines are Eq. (56), i.e. a single parameter fit $y = \alpha N^{-D_2}$ with α the fit parameter and D_2 independently determined from scaling of the moments (1) in direct space (for PRBM and RS) or taken from [68] (for 3DKR). See Appendix B for numerical procedure.

References:

M. Martinez *et al.*, PRR 3, 032044 (2021)

M. Martinez et al., scipost_202210_00061v1

CFS Contrast in the Large Size Limit at Finite Time

$t \ll \tau_H$

CFS arises from the **non-ergodicity** of the eigenstates and it no longer depends on N

Using Fourier transform from **k**-space to direct space, $\phi_n(\mathbf{k}) = \sum_{\mathbf{r}} \phi_n(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}}$, and some standard term-cancellation approximations, one can show that:

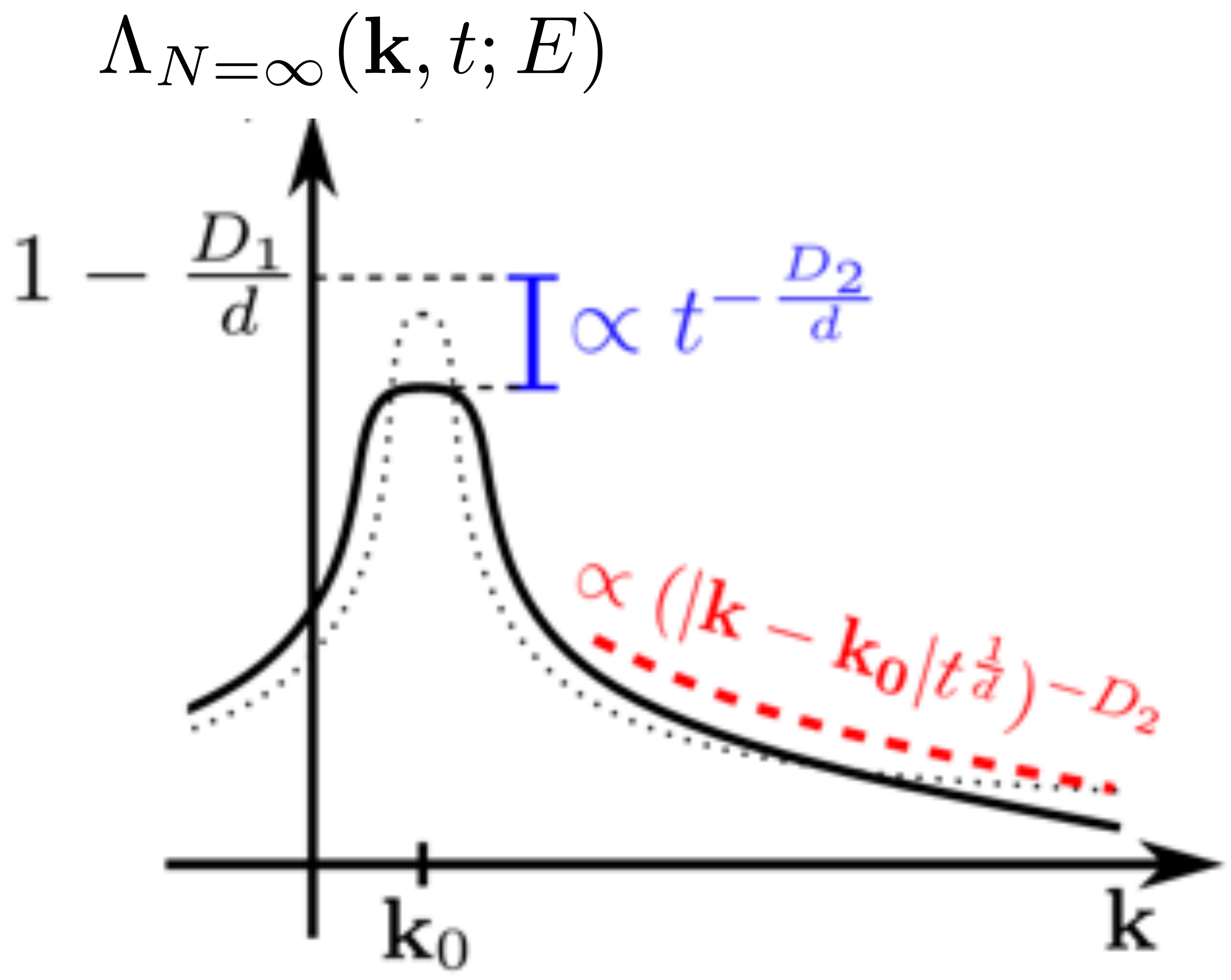
$$\Lambda_N(\mathbf{k}_0, t; E) = K_N(t; E) - P_{return}(t; E)$$

Spectral Form Factor

Return Probability

$$P_{return}(t; E) = \overline{\sum_{n,m,\mathbf{r}} e^{-i(E_n - E_m)t} |\phi_n(\mathbf{r})|^2 |\phi_m(\mathbf{r})|^2\, \delta(E - \frac{E_n + E_m}{2})}$$

4 eigenfunctions : It will encode D_2



$K_N(t; E) \to \kappa$ **Spectral Compressibility**

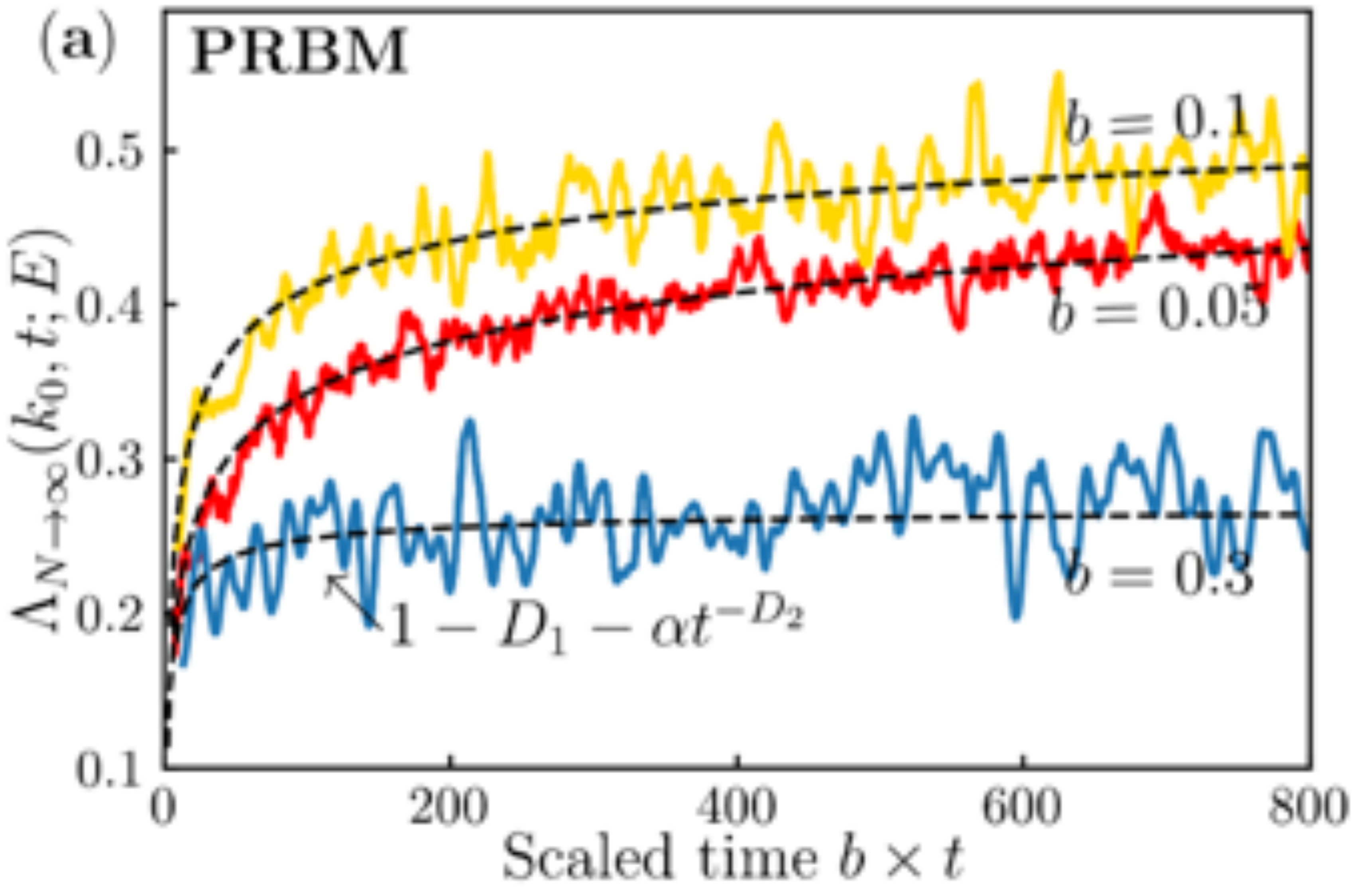
$\kappa = 1 - D_1/d$ **(B-G conjecture)**

$P_{return}(t; E) \sim t^{-D_2/d}$

Power-Law Random Banded Matrix Model

$N = 16384$

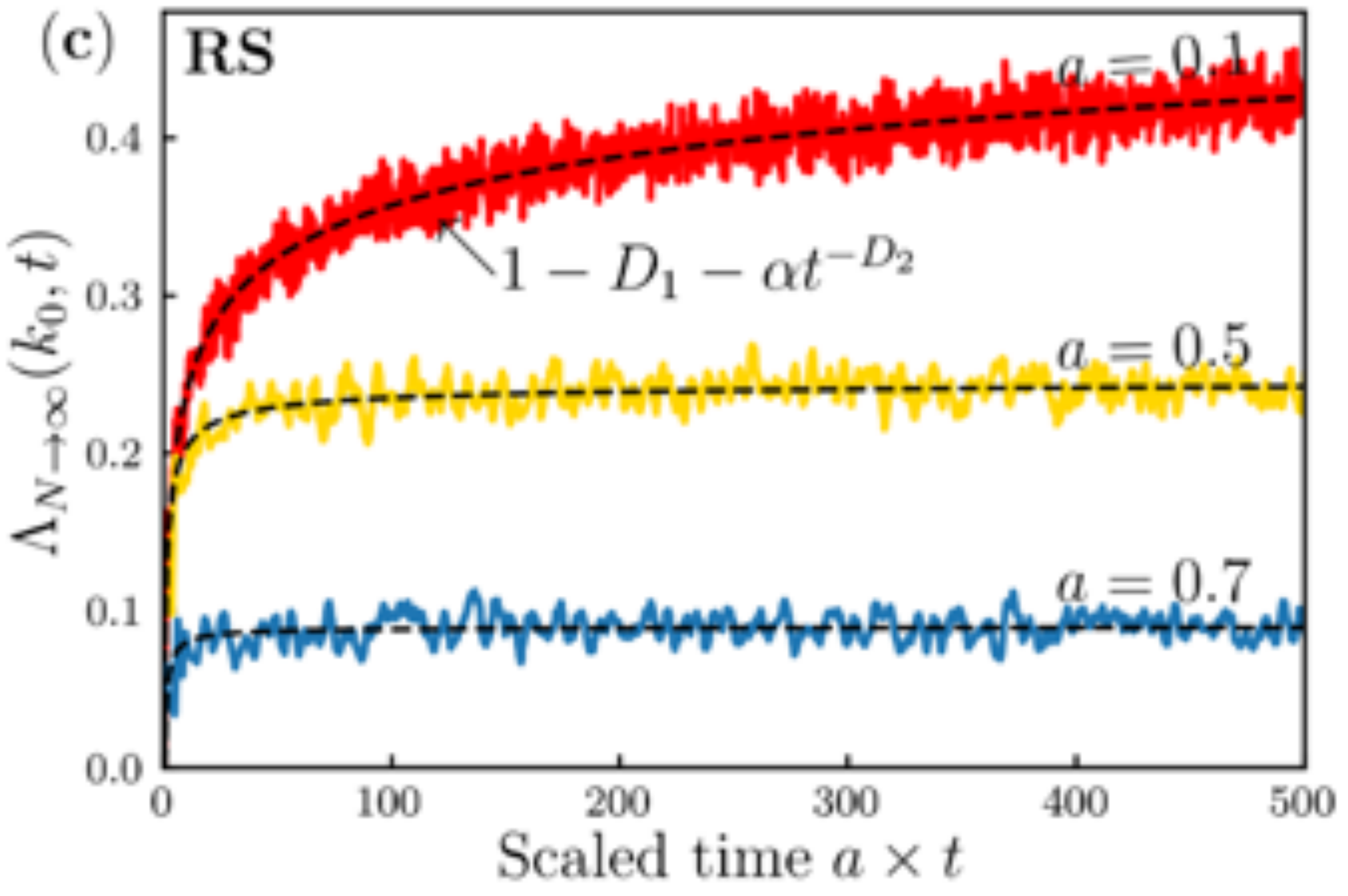
$N_{dis} = 1125$



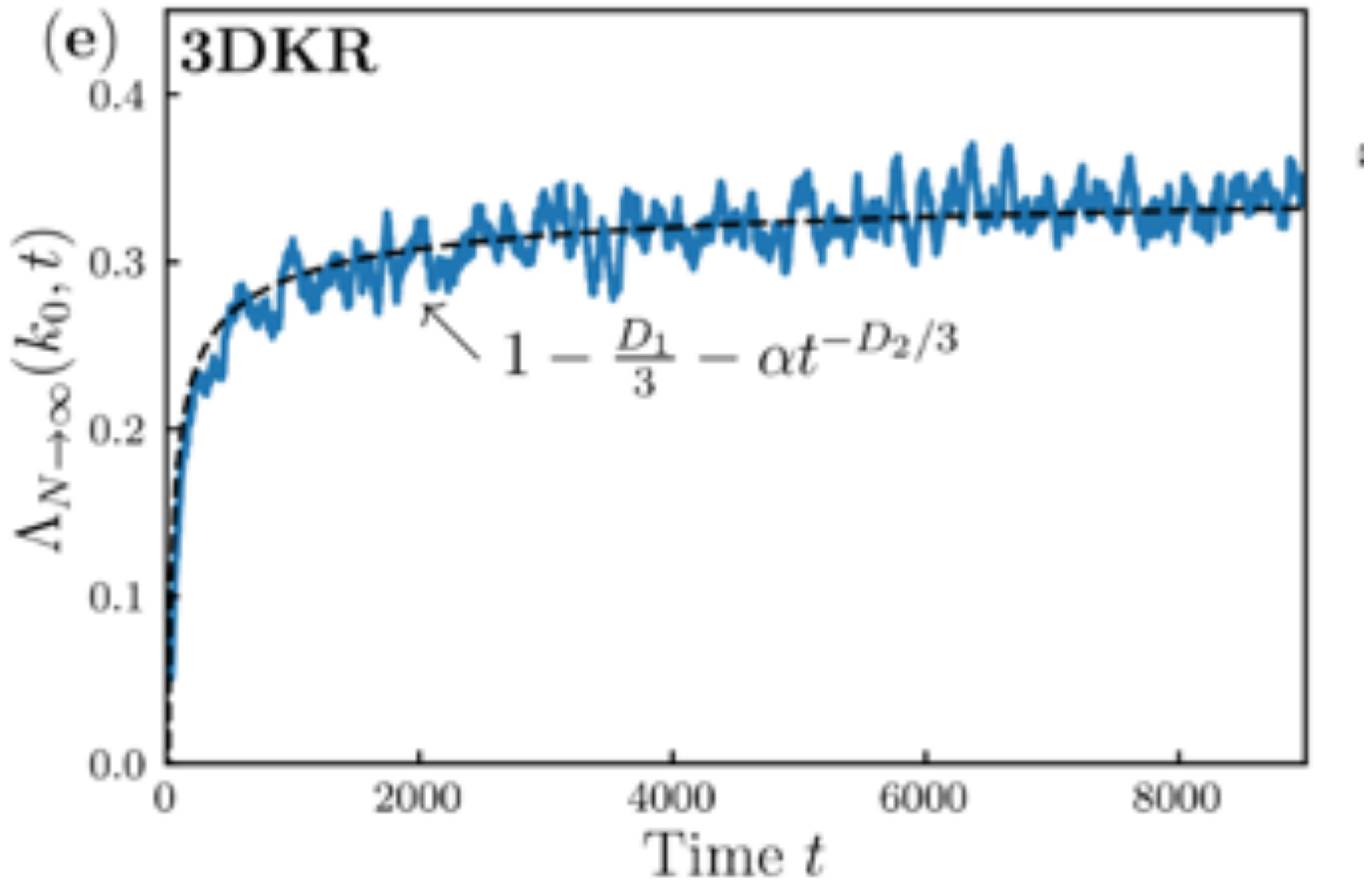
Ruijsenaars-Schneider Model

$N = 131072$

$N_{dis} = 3600$



3D Random Kicked Rotor Model



Conclusion & Perspectives

We have studied the CFS peak in critical disordered systems with multifractal eigenstates and showed that the CFS peak height time and system size dependence give access to fractal dimensions D_1 and D_2.

We have demonstrated that there exist two distinct dynamical regimes:

- **When $t \ll \tau_H$ (infinite system size limit)**

- CFS arises from the nonergodicity of the eigenstates.
- The CFS peak height reaches the compressibility $\kappa = 1 - D_1/d$ with a temporal power-law related to D_2.
- We also provide a full description of the shape of the CFS peak.

- **When $t \gg \tau_H$ (infinite time limit)**

- CFS is caused by the system boundaries.
- The height of CFS peak goes to 1 with a finite-size correction related to multifractal dimension D_2,
- We also provide an analysis of the CFS peak shape.

Question: Can we extract other fractal dimension from the dynamics of the peak shape?